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Off-shell $N = 2$ tensor supermultiplets

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Abstract: A multiplet calculus is presented for an arbitrary number n of $N = 2$ tensor supermultiplets. For rigid supersymmetry the known couplings are reproduced. In the superconformal case the target spaces parametrized by the scalar fields are cones over $(3n - 1)$-dimensional spaces encoded in homogeneous SU(2) invariant potentials, subject to certain constraints. The coupling to conformal supergravity enables the derivation of a large class of supergravity Lagrangians with vector and tensor multiplets and hypermultiplets. Dualizing the tensor fields into scalars leads to hypermultiplets with hyperkähler or quaternion-Kähler target spaces with at least n abelian isometries. It is demonstrated how to use the calculus for the construction of Lagrangians containing higher-derivative couplings of tensor multiplets. For the application of the c-map between vector and tensor supermultiplets to Lagrangians with higher-order derivatives, an off-shell version of this map is proposed. Various other implications of the results are discussed. As an example an elegant derivation of the classification of 4-dimensional quaternion-Kähler manifolds with two commuting isometries is given.

Keywords: Extended Supersymmetry, Differential and Algebraic Geometry, Supergravity Models.

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1. Introduction

The importance of off-shell methods for the construction of supersymmetric Lagrangians is well known. For $N = 2$ supersymmetry in four space-time dimensions the most relevant off-shell supermultiplets are the Weyl, the vector and the tensor supermultiplet. The Weyl supermultiplet comprises the fields of conformal supergravity, whereas the other two multiplets play the role of matter multiplets. The hypermultiplet does not constitute an off-shell multiplet, unless one introduces an infinite number of fields. This paper deals with $N = 2$ tensor supermultiplets whose off-shell formulation has a long history. In [1] the multiplet emerged as a submultiplet of off-shell $N = 2$ supergravity. Its transformation rules in a general superconformal background were presented in [2] and a locally superconformally invariant Lagrangian for a single tensor multiplet was written down in [3]. The latter enabled the derivation of an alternative minimal off-shell formulation of $N = 2$ supergravity.

In four space-time dimensions it is possible to dualize a rank-two tensor gauge field into a scalar field. In this way actions of tensor supermultiplets lead to corresponding supersymmetric actions for hypermultiplets. The resulting hypermultiplet target space will then have a group of abelian isometries induced by the gauge invariance of the tensor fields. In the case of rigid supersymmetry the hypermultiplets parametrize a hyperkähler space. In [4, 5] the $N = 1$ superspace formulation was used to classify, upon dualization, $4n$-dimensional hyperkähler metrics with n abelian isometries. The Lagrangians are encoded in terms of a function subject to certain partial differential equations, which can be elegantly written in terms of a contour integral depending on the tensor multiplet scalars. Furthermore a first $N = 2$ superspace formulation was presented in [6] in which this contour integral played a central role.

In the context of local $N = 2$ supersymmetry one is interested in superconformal tensor multiplets. The scalar fields then parametrize target spaces which are cones over a $(3n - 1)$-dimensional space. When coupling these supermultiplets, together with at least one vector multiplet and possible hypermultiplets, to conformal supergravity, the resulting theory is gauge equivalent to Poincaré supergravity coupled to matter fields. In this gauge equivalence the number of matter multiplets is reduced by two. This is so because part of the components belonging to the two, so-called compensating, supermultiplets correspond to superconformal gauge degrees of freedom. Upon gauge fixing the remaining components of these multiplets combine with the fields of the Weyl multiplet to constitute an off-shell multiplet of Poincaré supergravity. There is a certain freedom in choosing compensator multiplets, which leads to different off-shell versions of Poincaré supergravity. The more conventional one employs a compensating vector multiplet and a hypermultiplet, but the hypermultiplet can be replaced by a compensating tensor multiplet. These two choices do in certain cases lead to the same theory as the tensor fields can be dualized to scalar fields in which case the hypermultiplet target space becomes a quaternion-Kähler space. However, the dualization affects the off-shell supersymmetry structure.

When dualizing superconformal Lagrangians of tensor multiplets one obtains $4n$-dimensional hyperkähler cones [7]. The latter are cones over $(4n-1)$-dimensional 3-Sasakian spaces, which in turn are Sp(1) fibrations of $(4n-4)$-dimensional quaternion-Kähler spaces. In this context the gauge-fixing of the compensating degrees of freedom is known as a superconformal quotient and this quotient was extensively studied in [8]. The hyperkähler cones are encoded in so-called hyperkähler potentials and it turns out that there exits a similar real function for superconformal tensor multiplets that is homogeneous and SU(2) invariant. Just like the function exploited in [4, 5] it is subject to a set of partial differential equations. When applied to a single tensor supermultiplet acting as a compensator (in addition to a compensator vector supermultiplet), one recovers the results of [3] for pure supergravity with a tensor gauge field and local U(1) invariance. In this setting the tensor field does not describe dynamic degrees of freedom. For two tensor multiplets one finds pure supergravity with an additional matter multiplet, which contains two scalar and two tensor fields. Upon dualization of the tensor fields one obtains supergravity coupled to a single hypermultiplet whose target space defines a 4-dimensional quaternion-Kähler space. Solving the differential equations for the SU(2) invariant potential of the tensor formulation, one elegantly reproduces the general classification of the corresponding 4-dimensional metrics with two commuting Killing fields [9]. They include the metric of the so-called universal hypermultiplet as a special case.

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We should stress here that the above discussion is based on off-shell supermultiplets. When one is just interested in supersymmetric Lagrangians involving tensor fields, there are many more possibilities, as one can always dualize tensor gauge fields into scalar fields and, provided there are abelian isometries, vice versa. For a general discussion of $N = 2$ supersymmetric Lagrangians involving tensor and scalar fields, we refer to [10]. Naturally, these general Lagrangians are not encoded in a single function, unlike the Lagrangians derived through the superconformal quotient, but there are good reasons to believe that they can be derived from the same formalism by a series of dualizations [8].

The superconformal quotient for tensor supermultiplets was extensively discussed in [8] without paying attention to the details of their supergravity couplings. The first topic of this paper is therefore to extend the results of [3] to an arbitrary number of tensor supermultiplets. In the case of rigid supersymmetry, the results of this paper are completely in accord with [5]. It turns out that the coupling to conformal supergravity is straightforward in the present framework. The results can be used in the context of string compactifications where tensor fields arise naturally. Some of the results of this paper have already been exploited to derive string-loop corrected hypermultiplet metrics for type-II string theory compactified on a generic Calabi-Yau threefold [11]. Our work also has some overlap with, for example, that of [12] where dimensional reductions of five-dimensional supergravity theories are studied. For general gaugings the situation is less clear. It is known that magnetic background fluxes generically require the presence of tensor fields, which, however, acquire non-trivial mass terms [13 – 15]. Whether or not these tensor fields are in some way related to the tensor fields that are discussed here, is yet an open issue.

The results of this paper also enable the construction of higher-derivative actions for tensor supermultiplets. These actions contain terms of fourth order in space-time derivatives. We will demonstrate this by presenting one non-trivial example of such an action for a single tensor supermultiplet, encoded in a single function subject to differential constraints. To couple such an action to supergravity is straightforward and one has an additional option of including independent couplings with the Weyl multiplet or with vector multiplets in the form of a chiral background [16]. We intend to give a more complete presentation of these higher-derivative couplings elsewhere.

Vector supermultiplets can also have higher-derivative couplings. Also here we distinguish between vector multiplet couplings with the Weyl multiplet through a chiral background, and actions which contain *ab initio* higher-derivatives of the vector multiplet components themselves. The former are the ones relevant for the topological string [17] and have played an important role in the comparison between microscopic and macroscopic black hole entropy [18]. The latter are of the type studied, for example, in [19]. All these higher-order actions will undoubtedly contribute to the Wald entropy [20], which was crucial for obtaining agreement between microscopic and macroscopic black hole entropy at the subleading level in the limit of large charges.

It is clearly of interest to investigate on a par the higher-derivative couplings for both tensor and vector supermultiplets, as those are expected to be related by the so-called c-map [21]. Conventionally, the c-map is applied on the basis of actions that are at most quadratic in space-time derivatives [22 – 25]. In this way classical tensor (and thus hypermultiplet) moduli spaces that appear in compactifications of type-II strings can be determined from vector moduli spaces, as a result of T-duality. When considering actions with higher-order derivatives, also derivatives of auxiliary fields appear. Therefore we also study the definition of c-map for full off-shell supermultiplets, independent of the actions considered. The application of the c-map to higher-order derivative couplings was discussed in [26, 27] and in a recent paper [24].

This paper is organized as follows. In section 2 we discuss the tensor supermultiplets in the context of rigid supersymmetry. Following and extending the results of [3], we construct composite chiral multiplets in terms of tensor multiplet components. Subsequently we proceed to derive invariant actions. Furthermore we show how superconformally invariant actions are encoded in terms of a homogeneous SU(2) invariant potential, similar to the hyperkähler potentials for superconformal hypermultiplet Lagrangians. In section 3 we analyze the off-shell version of the c-map between vector and tensor multiplets and we present a nontrivial example of a supersymmetric action for a tensor supermultiplet involving higher-order derivatives. In section 4 we consider the coupling of tensor multiplets to conformal supergravity. In section 5 we discuss the superconformal quotient for Lagrangians involving tensor and vector multiplets and hypermultiplets to obtain Poincaré supergravity theories with tensor multiplets. To demonstrate the virtues of our formulation we consider the case of two tensor multiplets and evaluate the differential equations for the SU(2) invariant potential of the tensor formulation to obtain the classification of the corresponding 4-dimensional selfdual Einstein metrics with two commuting Killing fields. Finally some details of the superconformal calculus are presented in an appendix.

2. Rigid tensor multiplet couplings

The $N = 2$ tensor multiplet can be realized off-shell in a general superconformal background. In this section we consider the case of rigid supersymmetry in flat Minkowski space. The tensor supermultiplet is described in terms of a tensor gauge field $E_{\mu\nu}$, an SU(2) triplet of scalar fields L^{ij}, a doublet of Majorana spinors φ^i and an auxiliary complex scalar G. The supersymmetry transformation rules can be written as follows [2],

$$\begin{aligned}
\delta L_{ij} &= 2\,\bar\epsilon_{(i}\varphi_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\varphi^{l)}\,,\\
\delta\varphi^i &= \not{\partial} L^{ij}\,\epsilon_j + \varepsilon^{ij}\,\not{E}\,\epsilon_j - G\,\epsilon^i\,,\\
\delta G &= -2\bar\epsilon_i\,\not{\partial}\varphi^i\,,\\
\delta E_{\mu\nu} &= i\bar\epsilon^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij} - i\bar\epsilon_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij}\,,
\end{aligned} \tag{2.1}$$

where (anti)symmetrization is always defined with unit strength (unlike in [3]). Gamma matrices $\gamma^{\mu\nu\cdots}$ with multiple indices denote antisymmetrized products of gamma matrices in the usual fashion. We recall that ϵ^i and φ^i are positive-chirality spinors whose negative-chirality counterparts are denoted by ϵ_i and φ_i, respectively. Furthermore, $E^\mu = \frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu E_{\rho\sigma}$ is the invariant field strength of the tensor field. The scalar field L_{ij} satisfies a reality constraint, $L^{ij} = \varepsilon^{ik}\varepsilon^{jl}\,L_{kl}$. Complex conjugation is effected by raising and lowering of SU(2) indices, $i, j, k, \ldots$. Throughout this paper we use Pauli-Källén metric conventions.

2.1 Composite reduced chiral supermultiplets

Supersymmetric Lagrangians with at most two space-time derivatives can be constructed by making use of the observation that a tensor multiplet can couple linearly to a reduced chiral multiplet. The latter supermultiplet comprises a complex scalar X, an antisymmetric tensor $F_{\mu\nu}$, a (negative-chirality) spinor doublet Ω^i and a triplet of auxiliary scalars Y^{ij}. Its supersymmetry transformations are as follows,

$$
\begin{aligned}
\delta X &= \bar\epsilon^i\Omega_i\,,\\
\delta\Omega_i &= 2\,\partial\!\!\!/ X\,\epsilon_i + \tfrac{1}{2}\varepsilon_{ij}\,\gamma^{\mu\nu}F_{\mu\nu}\,\epsilon^j + Y_{ij}\,\epsilon^j\,,\\
\delta F^-_{\mu\nu} &= \tfrac{1}{2}\bar\epsilon_i\partial\!\!\!/\gamma_{\mu\nu}\Omega_j\,\varepsilon^{ij} - \tfrac{1}{2}\bar\epsilon^i\gamma_{\mu\nu}\partial\!\!\!/\Omega^j\,\varepsilon_{ij}\,,\\
\delta Y_{ij} &= 2\,\bar\epsilon_{(i}\,\partial\!\!\!/\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\,\partial\!\!\!/\Omega^{l)}\,.
\end{aligned}
\tag{2.2}
$$

Here $F^-_{\mu\nu}$ is the antiselfdual component of the tensor $F_{\mu\nu}$, whose complex conjugate equals $F^+_{\mu\nu}$. Because we are dealing with a reduced chiral multiplet, Y^{ij} satisfies a reality constraint, $Y^{ij} = \varepsilon^{ik}\varepsilon^{jl}\,Y_{kl}$ and $F_{\mu\nu}$ satisfies a Bianchi identity, $\partial_{[\mu}F_{\nu\rho]} = 0$. The latter can be solved (at least locally) so that $F_{\mu\nu}$ acquires the form $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$. The resulting vector supermultiplet can then be completed by specifying the transformation rule for W_μ,

$$
\delta W_\mu = \bar\epsilon_i\gamma_\mu\Omega_j\,\varepsilon^{ij} + \bar\epsilon^i\gamma_\mu\Omega^j\,\varepsilon_{ij}\,. \tag{2.3}
$$

As is well-known, there exists a non-abelian version of this multiplet which will, however, not be needed in what follows.

The supersymmetric coupling of a tensor to a reduced chiral multiplet takes the form,

$$
\mathcal{L} = X\,G + \bar X\,\bar G - \tfrac{1}{2}Y^{ij}\,L_{ij} + \bar\varphi^i\,\Omega_i + \bar\varphi_i\,\Omega^i - \tfrac{1}{4}i\,\varepsilon^{\mu\nu\rho\sigma}\,E_{\mu\nu}\,F_{\rho\sigma}\,. \tag{2.4}
$$

This expression can be used to derive supersymmetric Lagrangians for tensor multiplets, as was already demonstrated in [28, 3]. This derivation is based on the observation that one can construct a reduced chiral multiplet from tensor multiplet components. When substituting the components of this composite multiplet into (2.4) one obtains a supersymmetric Lagrangian for the tensor multiplet.

In order to construct n reduced chiral multiplets from n tensor multiplets, one must introduce a (real) function $\mathcal{F}_{I,J}(L)$ of the tensor multiplet scalars L^{ijI}, where we label the n tensor supermultiplets by upper indices $I, J, \ldots = 1, 2, \ldots, n$. The reduced chiral multiplet to which each tensor multiplet couples is then assigned a lower index I. The construction starts from the lowest component of the chiral multiplet, which is given by

$$
X_I = \mathcal{F}_{I,J}(L)\,\bar G^J + \mathcal{F}_{I,J,K}{}^{ij}(L)\,\bar\varphi_i{}^J\varphi_j{}^K\,, \tag{2.5}
$$

where

$$
\mathcal{F}_{I,J,Kij}(L) = \frac{\partial\mathcal{F}_{I,J}(L)}{\partial L^{ijK}}\,. \tag{2.6}
$$

We note that $\mathcal{F}_{I,J,Kij}$ satisfies the same reality constraint as the fields $L_{ij}{}^I$. Hence its SU(2) indices can be raised by complex conjugation, or alternatively, by contraction with epsilon

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tensors. Such quantities define real SU(2) vectors and their products satisfy certain product relations which reflect their decomposition in terms of irreducible SU(2) representations. We present two of them, which are used throughout this paper. The products of two such real vectors, L_{ij} and K_{ij}, satisfy

$$\begin{aligned} K_{ik}\, L^{jk} + K^{jk}\, L_{ik} &= \delta_i^j\, K_{kl}\, L^{kl}\,, \\ K_{ij}\, L_{kl} - K_{kl}\, L_{ij} &= \varepsilon_{ik}\, \varepsilon^{mn}\, (K_{lm}\, L_{nj} + K_{jm}\, L_{nl})\Big|_{(i,j)\,(k,l)}\,, \end{aligned} \tag{2.7}$$

where the right-hand side of the second equation is symmetrized in (i,j) and (k,l). These identities can be used with K_{ij} or L_{ij} equal to ${L_{ij}}^I$ or $\mathcal{F}_{I,J,Kij}$.

To ensure that we are dealing with a chiral multiplet the supersymmetry transformation of the composite field X_I has to be of the form (2.2). Up to terms cubic in the spinors ${\varphi_i}^I$ this imposes that the derivative $\mathcal{F}_{I,J,Kij}$ must be *symmetric* in (JK). The higher-order spinor terms require a second condition, namely,

$$\mathcal{F}_{I,J,K}{}^{ij}{}_{,L}{}^{kl}(L)\, (\bar\varphi_i{}^K \varphi_j{}^J)\, \varphi_k{}^L = 0\,, \tag{2.8}$$

where we defined

$$\mathcal{F}_{I,J,Kij,Lkl}(L) = \frac{\partial^2 \mathcal{F}_{I,J}(L)}{\partial L^{ijK}\, \partial L^{klL}}\;. \tag{2.9}$$

When $\mathcal{F}_{I,J,Kij,Lkl}(L)$ is symmetric in (jk) the cubic spinor term (2.8) vanishes. It is therefore guaranteed that we are dealing with a chiral multiplet once the following constraints are satisfied by the function $\mathcal{F}_{I,J}$,

$$\mathcal{F}_{I,J,Kij} = \mathcal{F}_{I,K,Jij}\,, \qquad \varepsilon^{jk}\, \mathcal{F}_{I,J,Kij,Lkl}(L) = 0\,. \tag{2.10}$$

As it turns out these constraints also suffice to ensure that we are dealing with a reduced chiral multiplet.

The function $\mathcal{F}_{I,J}$ has no particular symmetry in I and J. From the constraints (2.10) it follows that its derivatives with respect to the L^{ijK} are independently symmetric under the capital indices $J, K, \ldots$ and under the SU(2) indices $i, j, k, l, \ldots$. This motivates us to use an obvious notation $\mathcal{F}_{I,J_1\cdots J_{p+1} j_1\cdots j_{2p}}$ for the p-th multiple derivative, which is symmetric in both the $p+1$ indices $\{J\}$ and in the $2\,p$ indices $\{j\}$.

Henceforth we assume that the conditions (2.10) are satisfied. From the variation of (2.5) we determine the composite spinor field Ω_{iI} of the chiral multiplet,

$$\begin{aligned} \Omega_{i\,I} = {} & -2\, \mathcal{F}_{I,J}\, \partial\!\!\!/ \varphi_i{}^J + 2\, \mathcal{F}_{I,JKij}\, \bar G^J\, \varphi^{jK} - 2\, \mathcal{F}_{I,JK}{}^{kl}\, (\partial\!\!\!/ L_{ik}{}^J - \varepsilon_{ik} E\!\!\!/^J)\varphi_l{}^K \\ & + 2\, \mathcal{F}_{I,JKLij}{}^{kl}\, \varphi^{jL}\, (\bar\varphi_k{}^J \varphi_l{}^K)\,. \end{aligned} \tag{2.11}$$

The supersymmetry variation of $\Omega_{i\,I}$ yields the expressions for $Y_{ij\,I}$ and $F_{\mu\nu\,I}$, while all remaining variations correctly recombine into the derivative $\partial_\mu X_I$. The explicit expressions for the new fields read,

$$\begin{aligned} Y_{ij\,I} = {} & -2\, \mathcal{F}_{I,J}\, \partial^2 L_{ij}{}^J - 2\, \mathcal{F}_{I,JKij}\, (\bar G^J\, G^K + E_\mu{}^J\, E^{\mu K})\,, \\ & - 2\, \mathcal{F}_{I,JK}{}^{kl}\, (\partial_\mu L_{ik}{}^J\, \partial^\mu L_{jl}{}^K + 2\, \varepsilon_{k(i}\, \partial_\mu L_{j)l}{}^J\, E^{\mu K}) \end{aligned}$$

$$
\begin{aligned}
& - 2\,\mathcal{F}_{I,JKLij}{}^{kl}\,\bar\varphi_k{}^K \varphi_l{}^J\,G^L - 2\,\mathcal{F}_{I,JKLijkl}\,\bar\varphi^{kK}\varphi^{lJ}\,\bar G^L \\
& + 4\,(\mathcal{F}_{I,JKm(i}\,\bar\varphi^{mJ}\,\partial\!\!\!/\varphi_{j)}{}^K + \mathcal{F}_{I,JK}{}^{m(k}\bar\varphi_m{}^J \partial\!\!\!/\varphi^{l)K}\,\varepsilon_{ik}\,\varepsilon_{jl}) \\
& + 4\,\mathcal{F}_{I,JKLn(i}{}^{kl}\,\partial_\mu L_{j)k}{}^J\,\left(\bar\varphi^{nL}\gamma^\mu\varphi_l{}^K\right) \\
& - 4\,\mathcal{F}_{I,JKLn(i}{}^{kl}\,\varepsilon_{j)k}\,\left(\bar\varphi^{nL}\,E\!\!\!/^J\,\varphi_l{}^K\right) \\
& - 2\,\mathcal{F}_{I,JKLMijmn}{}^{kl}\,\bar\varphi_k{}^J\varphi_l{}^K\,\bar\varphi^{mL}\varphi^{nM}\,, \\
F_{\mu\nu\,I} =\; & - 2\,\mathcal{F}_{I,JK}{}^{mn}\,\partial_{[\mu}L_{mk}{}^J\,\partial_{\nu]}L_{nl}{}^K\,\varepsilon^{kl} \\
& - 4\,\partial_{[\mu}\Big(\mathcal{F}_{I,J}\,E_{\nu]}{}^J + \mathcal{F}_{I,JKki}\,\bar\varphi^{kJ}\gamma_{\nu]}\varphi_j{}^K\,\varepsilon^{ij}\Big)\,. \qquad (2.12)
\end{aligned}
$$

The results can be compared to the corresponding ones given in [3]. In order that we are dealing with a single reduced chiral superfield for given index I, it is important that $\mathcal{F}_{I,J}$ is a real function. This enables the use of identities such as (2.7). These identities and (2.10) are used throughout the calculation. The Bianchi identity holds for $F_{\mu\nu\,I}$, although the second term proportional to $\partial_{[\mu}L\,\partial_{\nu]}L$ is somewhat subtle. By virtue of (2.10) the contribution of this term, $\partial_{[\mu}F_{\nu\rho]} \propto \mathcal{F}_{I,JKLijkl}\varepsilon_{mn}\,\partial_{[\mu}L^{ijJ}\,\partial_\nu L^{kmK}\,\partial_{\rho]}L^{lnL}$, vanishes so that $F_{\mu\nu}$ is closed. However, $F_{\mu\nu}$ is not exact in the sense that it cannot be written as the curl of a manifestly SU(2) invariant quantity. We will exhibit this below.

Let us now discuss the constraints (2.10). To analyze their implications, we decompose the field L^{ijI} into a real field x^I and a complex field v^I according to,

$$
L^{12\,I} \equiv \tfrac{1}{2}\mathrm{i}\,x^I\,, \qquad L^{11\,I} \equiv v^I\,, \qquad (2.13)
$$

so that $L^I_{ij}L^{ij\,J} = \frac{1}{2}x^Ix^J + 2\,v^{(I}\bar v^{J)}$. The constraints (2.10) then take the following form,[1]

$$
\begin{aligned}
& \frac{\partial\mathcal{F}_{I,J}}{\partial x^K} = \frac{\partial\mathcal{F}_{I,K}}{\partial x^J}\,, \qquad \frac{\partial\mathcal{F}_{I,J}}{\partial v^K} = \frac{\partial\mathcal{F}_{I,K}}{\partial v^J}\,, \\
& \frac{\partial^2\mathcal{F}_{I,J}}{\partial x^K\partial x^L} + \frac{\partial^2\mathcal{F}_{I,J}}{\partial v^K\partial\bar v^L} = 0\,. \qquad (2.14)
\end{aligned}
$$

The last equation, which simply follows from $\mathcal{F}_{I,JKLij}{}^{ij} = 0$, contains the SU(2) invariant Laplacian,

$$
\tfrac{1}{2}\,\varepsilon_{ik}\varepsilon_{jl}\,\frac{\partial^2}{\partial L_{ij}{}^I\,\partial L_{kl}{}^J} = \frac{\partial^2}{\partial x^I\,\partial x^J} + \frac{\partial^2}{\partial v^{(I}\,\partial\bar v^{J)}}\,. \qquad (2.15)
$$

As a consequence of the first equation of (2.14), $\mathcal{F}_{I,J}$ can be expressed as a derivative of a new function $\mathcal{F}_I$ which is, however, still constrained,

$$
\begin{aligned}
& \mathcal{F}_{I,J} = \frac{\partial\mathcal{F}_I}{\partial x^J}\,, \qquad \frac{\partial^2\mathcal{F}_I}{\partial x^J\partial v^K} = \frac{\partial^2\mathcal{F}_I}{\partial x^K\partial v^J}\,, \\
& \frac{\partial^2\mathcal{F}_I}{\partial x^J\partial x^K} + \frac{\partial^2\mathcal{F}_I}{\partial v^J\partial\bar v^K} = 0\,. \qquad (2.16)
\end{aligned}
$$

The last equation of (2.16) was determined by integrating the last equation of (2.14) which leaves a real function on the right-hand side that does not depend on x. However, differentiation with respect to v^L (or $\bar v^L$) yields a function symmetric in (J,L) (or (K,L))

[1] Derivatives with respect to $L_{ij}{}^I$ are defined by $\frac{\partial}{\partial L_{ij}{}^J}\,L_{kl}{}^I = \frac{1}{2}\left(\delta^i_k\,\delta^j_l + \delta^i_l\,\delta^j_k\right)\,\delta^I_J$, so that $\delta L_{ij}{}^I\partial/\partial L^I_{ij} = \delta x^I\partial/\partial x^I + \delta v^I\partial/\partial v^I + \delta\bar v^I\partial/\partial\bar v^I$.

which implies that the right-hand side can be written as the $\partial^2/\partial v^J \partial\bar v^K$ derivative of some function of v and $\bar v$. As $\mathcal{F}_I$ is defined up to an x-independent function, the latter can be absorbed into $\mathcal{F}_I$.

With these results we can now exhibit that the expression for $F_{\mu\nu\, I}$ given in (2.12) takes indeed the form of a curl,

$$\mathcal{F}_{I,JK}{}^{ij}\,\partial_{[\mu}L_{ik}{}^J\,\partial_{\nu]}L_{jl}{}^K\,\varepsilon^{kl} = \mathrm{i}\,\partial_{[\mu}\Big(\frac{\partial\mathcal{F}_I}{\partial\bar v^J}\,\partial_{\nu]}\bar v^J - \frac{\partial\mathcal{F}_I}{\partial v^J}\,\partial_{\nu]}v^J\Big)\,, \tag{2.17}$$

so that the Bianchi identity is manifestly satisfied.

Let us close with two examples which lead to Lagrangians (constructed according to the procedure outlined in the next subsection) that are both dual to non-interacting hypermultiplets. One concerns the simple example where $\mathcal{F}_{I,J} = \delta_{IJ}$ is L-independent. This example trivially satisfies the constraints (2.10). One possible expression for $\mathcal{F}_I$ takes the form,

$$\mathcal{F}_I = x^I + c_{IJ}\,v^J + \bar c_{IJ}\,\bar v^J\,, \tag{2.18}$$

with c_{IJ} some complex constants. A second example is based on the conformal tensor multiplet introduced in [3, 4], where $\mathcal{F}_{I,J} = \delta_{IJ}\,(L^I)^{-1}$ with $L^I = \sqrt{L_{ij}{}^I\,L^{ijI}}$, so that, for I, J, K, L equal,

$$\mathcal{F}_{I,JKij}(L) = -\frac{L_{ij}{}^I}{(L^I)^3}\,,\qquad \mathcal{F}_{I,JKLijkl}(L) = \frac{3\,L_{ij}{}^I\,L_{kl}{}^I + (L^I)^2\,\varepsilon_{i(k}\varepsilon_{l)j}}{(L^I)^5}\,, \tag{2.19}$$

which satisfies the constraints (2.10). A corresponding expression for $\mathcal{F}_I$ is given by

$$\mathcal{F}_I = \sqrt{2}\ln\Big[x^I + \sqrt{x^I x^I + 4\,v^I\bar v^I}\Big] - \tfrac{1}{2}\sqrt{2}\ln\Big[4\,v^I\bar v^I\Big]\,. \tag{2.20}$$

2.2 Supersymmetric tensor multiplet actions

We now proceed to give the rigidly supersymmetric tensor multiplet Lagrangian obtained by substituting the composite fields (2.5), (2.11) and (2.12) into the density formula (2.4). Up to total derivatives the Lagrangian equals

$$\begin{aligned}\mathcal{L} = {}& F_{IJ}\Big[-\tfrac{1}{2}\partial_\mu L_{ij}{}^I\,\partial^\mu L^{ijJ} + E^I_\mu\,E^{\mu J} - (\bar\varphi^{iI}\,\partial\!\!\!/\varphi_i{}^J + \bar\varphi_i{}^I\,\partial\!\!\!/\varphi^{iJ}) + G^I\,\bar G^J\Big]\\ &+ \tfrac{1}{2}ie^{-1}\varepsilon^{\mu\nu\rho\sigma}\,F_{IJK}{}^{ij}\,E^I_{\mu\nu}\,\partial_\rho L_{ik}{}^J\,\partial_\sigma L_{jl}{}^K\,\varepsilon^{kl}\\ &- F_{IJK}{}^{ij}\Big[\bar\varphi^{kI}\,\partial\!\!\!/ L_{jk}{}^J\,\varphi_i{}^K - G^I\,\bar\varphi_i{}^J\varphi_j{}^K\Big]\\ &- F_{IJK\,ij}\Big[\bar\varphi_k{}^I\,\partial\!\!\!/ L^{jkJ}\,\varphi^{iK} - \bar G^I\,\bar\varphi^{iJ}\varphi^{jK}\Big]\\ &+ 2\,F_{IJK}{}^{ij}\,\varepsilon_{ki}\,\bar\varphi^{kI}\,E\!\!\!/^J\,\varphi_j{}^K\\ &+ F_{IJKL\,ij}{}^{kl}\,\bar\varphi_k{}^I\,\varphi_l{}^J\,\bar\varphi^{iK}\varphi^{jL}\,,\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}F_{IJ} &= 2\,\mathcal{F}_{(I,J)} + L^{ijK}\,\mathcal{F}_{K,IJij}\,,\\ F_{IJK}{}^{ij} &= 3\,\mathcal{F}_{(I,JK)}{}^{ij} + L^{klL}\mathcal{F}_{L,IJKkl}{}^{ij} = \frac{\partial F_{IJ}}{\partial L_{ij}{}^K}\,,\end{aligned}$$

$$
F_{IJKL\,ij}{}^{kl} = 4\,\mathcal{F}_{(I,JKL)ij}{}^{kl} + L^{mnM}\mathcal{F}_{M,IJKLmnij}{}^{kl} = \frac{\partial^2 F_{IJ}}{\partial L^{ijK}\partial L_{kl}{}^{L}}\,. \tag{2.22}
$$

We note that the tensor gauge field always appears in form of the covariant field strength E^μ, with the exception of the second line proportional to $\varepsilon^{\mu\nu\rho\sigma}$. This term is nevertheless invariant under tensor gauge transformations, up to a total derivative, owing to the Bianchi identity satisfied by the L-dependent terms. In the basis (2.13), this term can be rewritten in terms of the tensor field strength after partial integration, as we shall discuss shortly (*c.f.* (2.28)).

The Lagrangian is encoded in the function F_{IJ} and its derivatives. Making use of (2.16), the functions F_{IJ} can be written as follows,

$$
\begin{aligned}
F_{IJ} &= \frac{\partial\mathcal{F}_I}{\partial x^J} + \frac{\partial\mathcal{F}_J}{\partial x^I} + x^K\frac{\partial^2\mathcal{F}_K}{\partial x^I\partial x^J} + v^K\frac{\partial^2\mathcal{F}_K}{\partial x^I\partial v^J} + \bar v^K\frac{\partial^2\mathcal{F}_K}{\partial x^I\partial\bar v^J}\,,\\
&= \frac{\partial}{\partial x^I}\Big[\mathcal{F}_J + x^K\frac{\partial\mathcal{F}_K}{\partial x^J} + v^K\frac{\partial\mathcal{F}_K}{\partial v^J} + \bar v^K\frac{\partial\mathcal{F}_K}{\partial\bar v^J}\Big]\,.
\end{aligned} \tag{2.23}
$$

This expression is symmetric in (I,J). Thus the terms inside the bracket are equal to the x^J-derivative of another function. Therefore F_{IJ} can be written as the second x-derivative of some unknown function $F(x,v,\bar v)$. Integrating (2.23) yields the first derivative of F,

$$
\frac{\partial F}{\partial x^J} = \mathcal{F}_J + x^K\frac{\partial\mathcal{F}_K}{\partial x^J} + v^K\frac{\partial\mathcal{F}_K}{\partial v^J} + \bar v^K\frac{\partial\mathcal{F}_K}{\partial\bar v^J}\,, \tag{2.24}
$$

up to an x-independent function which we set to zero. Next we evaluate $\partial^2F/\partial v^I\times\partial x^J$ and establish its symmetry in (I,J) from (2.16). Furthermore we verify that

$$
\frac{\partial}{\partial x^I}\Big[\frac{\partial^2F}{\partial x^J\partial x^K} + \frac{\partial^2F}{\partial v^J\partial\bar v^K}\Big] = 0\,, \tag{2.25}
$$

making use again of (2.16). By following the same argument as below (2.16), one then establishes the existence of a function F subject to the equations,

$$
\frac{\partial^2F}{\partial x^I\partial v^J} = \frac{\partial^2F}{\partial x^J\partial v^I}\,,\qquad \frac{\partial^2F}{\partial x^I\partial x^J} + \frac{\partial^2F}{\partial v^I\partial\bar v^J} = 0\,. \tag{2.26}
$$

The Lagrangian is thus encoded in functions $F(x,v,\bar v)$, with

$$
F_{IJ} = \frac{\partial^2F}{\partial x^I\partial x^J}\,, \tag{2.27}
$$

and $F(x,v,\bar v)$ subject to the conditions (2.26). This result is entirely consistent with the results derived in [4, 5], where it was shown how to express the function $F(x,v,\bar v)$ in terms of a contour integral.

Using the above relations we derive, along the same lines as in (2.17), the relation,

$$
F_{IJK}{}^{ij}\,\partial_{[\mu}L_{ik}{}^J\,\partial_{\nu]}L_{jl}{}^K\,\varepsilon^{kl} = \mathrm{i}\,\partial_{[\mu}\Big(\frac{\partial^2F}{\partial x^I\,\partial\bar v^J}\,\partial_{\nu]}\bar v^J - \frac{\partial^2F}{\partial x^I\,\partial v^J}\,\partial_{\nu]}v^J\Big)\,. \tag{2.28}
$$

This result is needed when dualizing the tensor fields to scalars. In that case the supersymmetry is no longer realized off shell. One introduces a new set of fields, y_I, which act as

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Lagrange multipliers to impose the Bianchi identity on the tensor field strength. Adding the term $y_I\,\partial_\mu E^{\mu I}$ to the Lagrangian and integrating out the $E^{\mu I}$, one obtains an action for hypermultiplets. A natural set of complex variables then consists of the complex fields v^I and w_I. The latter are defined by [8]

$$w_I = \frac{1}{2}\Big(\mathrm{i}y_I + \frac{\partial F}{\partial x^I}\Big)\,. \tag{2.29}$$

In terms of these fields the kinetic term of the scalar fields reads,

$$\mathcal{L} = -F_{IJ}\,\partial_\mu v^I\partial^\mu\bar v^J - F^{IJ}\Big(\partial_\mu w_I - \frac{\partial^2 F}{\partial x^I v^K}\,\partial_\mu v^K\Big)\Big(\partial^\mu \bar w_J - \frac{\partial^2 F}{\partial x^J \bar v^L}\,\partial^\mu \bar v^L\Big)\,, \tag{2.30}$$

where F^{IJ} is the inverse of F_{IJ}.

For completeness we present the functions $F(x,v,\bar v)$ corresponding to the two examples (2.18) and (2.20), respectively,

$$\begin{aligned}
F(x,v,\bar v) &= \sum_I \Big\{(x^I)^2 - 2\,v^I\bar v^I\Big\}\,,\\
F(x,v,\bar v) &= \sqrt{2}\sum_I \Big\{x^I \ln\Big[x^I + \sqrt{(x^I)^2 + 4\,v^I\bar v^I}\Big] + \tfrac{1}{2}(1-x^I)\ln\Big[4\,v^I\bar v^I\Big]\\
&\qquad\qquad -\sqrt{(x^I)^2+4\,v^I\bar v^I}\,\Big\}\,.
\end{aligned} \tag{2.31}$$

2.3 Superconformal actions and tensor and hyperkähler cones

So far our analysis was completely general and we did not insist on any additional invariance beyond $N = 2$ supersymmetry. However, a tensor supermultiplet can be assigned to a representation of the full $N = 2$ superconformal algebra and the function $\mathcal{F}_{I,J}$ can be chosen such that the composite chiral supermultiplet constitutes also a superconformal representation. By substituting the superconformally invariant composite chiral multiplets into the density formula these symmetries carry over to the Lagrangian. The class of superconformal actions is encoded by functions $\mathcal{F}_{I,J}$ that satisfy the additional restriction,

$$\mathcal{F}_{I,JKik}\,L^{kjK} = -\tfrac{1}{2}\delta_i{}^j\,\mathcal{F}_{I,J}\,. \tag{2.32}$$

This condition, which will be derived in section 4, implies that $\mathcal{F}_{I,J}$ is a homogeneous function of the L^{ijI} of degree -1 that is invariant under the SU(2) R-symmetry. It is easy to see that the function F_{IJ} that appears in the Lagrangian (2.21), is thus also homogeneous of degree -1 and SU(2) invariant. Upon contraction with $L_{jm}{}^J$ one proves another useful result,

$$\mathcal{F}_{I,JKij}\,L_{kl}{}^J\,L^{klK} = -\mathcal{F}_{I,J}\,L_{ij}{}^J\,, \tag{2.33}$$

which is needed later on. The same result applies to F_{IJ}.

The constraint (2.32) implies that the function $\mathcal{F}_I$ can be restricted to a homogeneous function of zeroth degree, but, in general, it is only invariant under a U(1) subgroup of the SU(2) R-symmetry. The superconformal constraints on the function $F(x,v,\bar v)$, which is a homogeneous function of degree +1, were extensively analyzed in [8]. For convenience, we summarize the conditions on the function F_{IJ}. In the general case we have the constraints,

$$F_{IJK}{}^{ij} = F_{(IJK)}{}^{ij}\,, \qquad F_{IJKL}{}^{i[jk]l} = 0\,. \tag{2.34}$$

For conformally invariant Lagrangians there is the additional constraint,

$$F_{IJKik}\,L^{kjK} = -\tfrac{1}{2}\delta_i{}^j\,F_{IJ}\,. \tag{2.35}$$

When keeping one of the components in the triplet $L_{ij}{}^I$, x^I say, fixed the remaining complex components v^I parameterize a Kähler space whose corresponding Kähler potential is equal to the function $-F(x,v,\bar v)$. In the conformally invariant case a similar potential exists for the target space parametrized by the $L_{ij}{}^I$, which is defined by the SU(2) invariant expression,

$$\chi_{\rm tensor}(L) = 2\,F_{IJ}\,L^{ijI}L_{ij}{}^J\,, \tag{2.36}$$

and is a homogeneous function of degree +1. This potential is closely related to the so-called hyperkähler potential that plays a similar role in the hypermultiplet case. To see this we first note that its derivative with respect to L^I is equal to the homothetic vector,

$$\frac{\partial\chi_{\rm tensor}(L)}{\partial L_{ij}{}^I} = 2\,F_{IJ}L^{ijJ}\,. \tag{2.37}$$

This vector generates the scale transformations on $L_{ij}{}^I$ with scaling weight equal to 2. Furthermore we establish that the metric F_{IJ} is related to the second-order derivative of the potential, according to

$$\varepsilon_{kl}\,\frac{\partial^2\chi_{\rm tensor}(L)}{\partial L_{ik}{}^I\,\partial L_{jl}{}^J} = 2\,F_{IJ}(L)\,\varepsilon^{ij}\,. \tag{2.38}$$

This implies that the $3n$-dimensional target space parametrized by the $L_{ij}{}^I$ is a cone over a $(3n-1)$-dimensional space. One can show that the potential $\chi_{\rm tensor}$ fully encodes the superconformal theories of tensor supermultiplets. From it the function $F(x,v,\bar v)$ can be determined by integration. In section 5 the role of $\chi_{\rm tensor}$ will be clarified further.

To elucidate the above, let us formulate it in terms of the variables v^I, $\bar v^I$ and x^I. Using (2.27) one establishes the following identity,

$$\chi_{\rm tensor}(L) = F_{IJ}(x^Ix^J + 4\,v^I\bar v^J) = -F(v,\bar v,x) + x^I\,\frac{\partial F(x,v,\bar v)}{\partial x^I}\,, \tag{2.39}$$

where we made use of the various identities for derivatives of the function $F(x,v,\bar v)$. The right-hand side of (2.39) coincides with the expression for the hyperkähler potential given in [8] for the hyperkähler cones that one obtains upon dualizing the tensor fields to scalars. Here the x^I are expressed in terms of the coordinates $w_I+\bar w_I$ given in (2.29). Obviously the hyperkähler potential $\chi_{\rm hyper}(w_I,\bar w_I,v^I,\bar v^I)$ and the function $F(x,v,\bar v)$ are related by a Legendre transform.

The formalism of this paper makes it straightforward to incorporate the coupling of tensor supermultiplets to conformal supergravity. In [3] this was demonstrated for a single tensor supermultiplet and in section 4 we will generalize this result to n tensor supermultiplets. Before turning to this topic we first discuss a number of other features in the next section.

3. Off-shell c-map and higher-derivative actions

We have already stressed the importance of dealing with off-shell supermultiplets which offer many technical advantages. In the first subsection 3.1 we will illustrate this once more by introducing the c-map between off-shell tensor and vector supermultiplets, outside the context of specific supersymmetric actions. The fact that the c-map can be defined in this way is crucial for its application to higher-derivative actions, where the existence of an off-shell formulation is almost imperative. Without off-shell multiplets higher-derivative actions can only be constructed by an infinite series of iterations. Therefore we also briefly consider the construction of higher-derivative couplings of tensor supermultiplets in a second subsection 3.2. The coupling to supergravity will be the subject of later sections, but we will already present the extra bosonic terms that are generated in the coupling to supergravity.

3.1 The off-shell c-map

As is well known, four-dimensional vector- and hypermultiplet actions are related to each other via the so-called c-map. Originally [21] this map was constructed by performing a dimensional reduction of the four-dimensional action on a circle and dualizing the three-dimensional vector field to a scalar. Because these operations do not affect supersymmetry, the vector multiplets are converted into hypermultiplets, so that one will be dealing with two hypermultiplet sectors. Interchanging the two sectors and lifting back to a four-dimensional action (assuming that the initial hypermultiplet sector is itself in the image of the c-map) yields the desired map between vector- and hypermultiplet sectors in four dimensions.

A more natural way to define the c-map is by comparing a dimensionally reduced vector supermultiplet to a dimensionally reduced tensor supermultiplet. Indeed it is immediately clear that there exists a close relationship between the off-shell degrees of freedom. When reducing on a circle in the 3-direction, the space-time coordinate vector x^μ decomposes into a three-dimensional space-time vector $x^{\hat\mu}$ ($\hat\mu = 0,1,2$) and a single coordinate x^3 which will be shrunk to a point so that the fields become x^3-independent. In this way the bosonic fields of the tensor multiplet decompose according to,

$$\Big\{L_{ij}, E^\mu, G, \bar G\Big\} \longrightarrow \Big\{L_{ij}, E^{\hat\mu}, E^3, G, \bar G\Big\}\,, \tag{3.1}$$

where $E^{\hat\mu}$ is a divergence-free vector field. Likewise the bosonic fields of the (abelian) vector multiplet decompose according to,

$$\Big\{X, \bar X, F_{\mu\nu}, Y^{ij}\Big\} \longrightarrow \Big\{X, \bar X, F_{\hat\mu 3}, F_{\hat\mu\hat\nu}, Y^{ij}\Big\} \sim \Big\{X, \bar X, W_3, F^{\hat\mu}, Y^{ij}\Big\}\,. \tag{3.2}$$

In the last step we made use of the Bianchi identity satisfied by $F_{\mu\nu}$, which implies that $F_{\hat\mu\hat\nu}$ is equivalent to a divergence-free three-vector $F^{\hat\mu} = \frac{1}{4}\mathrm{i}\varepsilon^{\hat\mu\hat\nu\hat\rho}F_{\hat\nu\hat\rho}$ and that $F_{\hat\mu 3}$ can be written as the derivative of a scalar field W_3. Hence the two multiplets are very similar. They both comprise a single divergence-free vector, three physical scalars and three auxiliary scalars, and they have the same number of fermionic degrees of freedom. Both divergence-free vectors can be expressed in terms of a vector potential which coincides (up to a gauge transformation) with $E_{\hat{\mu}3}$ and $W_{\hat{\mu}}$, respectively.

The relation between the two supermultiplets becomes even more striking upon realizing that the R-symmetry group, the relativistic automorphism group of the supersymmetry algebra, which equals $\mathrm{SU}(2)\times\mathrm{U}(1)$ in four space-time dimensions, is extended to $\mathrm{SU}(2)\times\mathrm{SU}(2)$ in three space-time dimensions. Since the action of the U(1) subgroup is known on the four-dimensional fields, it is not difficult to deduce the representation content of the fields in three dimensions. Obviously, the fermionic fields must transform according to the $(2,2)$ representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, while the triplets Y_{ij} and L_{ij} transform according to the $(3,1)$ representation. Finally the triplets $\{X,\bar{X},W_3\}$ and $\{G,\bar{G},E^3\}$ must transform according to the $(1,3)$ representation. Obviously the two off-shell multiplets are the same and only differ in their identification with the SU(2) factors of the R-symmetry group.

The above conclusions are confirmed by an evaluation of the supersymmetry transformation rules in the three-dimensional context, following [29]. First we define gamma matrices $\hat{\gamma}^{\hat{\mu}}$ that are appropriate for the three-dimensional theory,

$$\hat{\gamma}^{\hat{\mu}} = \gamma^{\hat{\mu}}\,\tilde{\gamma}\,, \tag{3.3}$$

where $\tilde{\gamma} = -\mathrm{i}\gamma^3\gamma^5$ is an hermitean matrix whose square is equal to the identity matrix. The product $\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2$ is proportional to the identity matrix. The hermitean matrices $\tilde{\gamma}$, γ^3 and γ^5 commute with the $\hat{\gamma}^{\hat{\mu}}$ and constitute the generators of an su(2) algebra that is related to the second SU(2) factor of the R-symmetry group in three dimensions. Obviously, $\mathrm{i}\gamma^5$, the U(1) R-symmetry generator of the four-dimensional theory is contained. The second set of SU(2) transformations mixes spinors of different chirality. On the supersymmetry parameters with (anti)chiral components ϵ^i (ϵ_i), the 'hidden' SU(2) transformations act according to [29],

$$\delta\epsilon^i = -\tfrac{1}{2}\mathrm{i}\alpha\,\epsilon^i + \tfrac{1}{2}\beta\,\varepsilon^{ij}\gamma^3\,\epsilon_j\,, \qquad \delta\epsilon_i = \tfrac{1}{2}\mathrm{i}\alpha\,\epsilon_i + \tfrac{1}{2}\bar{\beta}\,\varepsilon_{ij}\gamma^3\,\epsilon^j\,, \tag{3.4}$$

where α is a real parameter associated with the chiral U(1) R-symmetry in four dimensions and β is complex. It is straightforward to verify that the above transformations generate a group SU(2) that commutes with the four-dimensional SU(2) R-symmetry group.

Now we present the three-dimensional supersymmetry transformations upon the reduction to three space-time dimensions, which readily follow from (2.1), (2.2) and (2.3), and identify the R-symmetry transformations. The result for the tensor multiplet reads as follows[2]

$$\begin{aligned}
\delta L_{ij} &= 2\,\mathrm{i}\,\bar\epsilon_{(i}\gamma^3\varphi_{j)} - 2\,\mathrm{i}\,\varepsilon_{ik}\,\varepsilon_{jl}\,\bar\epsilon^{(k}\gamma^3\varphi^{l)}\,,\\
\delta E_{\hat\mu 3} &= \mathrm{i}\bar\epsilon^i\,\hat\gamma_{\hat\mu}\,\gamma^3\,\varphi^j\,\varepsilon_{ij} - \mathrm{i}\bar\epsilon_i\,\hat\gamma_{\hat\mu}\,\gamma^3\,\varphi_j\,\varepsilon^{ij}\,,\\
\delta\varphi^i &= \mathrm{i}\,\hat{\partial\!\!\!/} L^{ij}\,\gamma^3\,\epsilon_j + \mathrm{i}\varepsilon^{ij}\,E^{\hat\mu}\hat\gamma_{\hat\mu}\gamma^3\,\epsilon_j + \varepsilon^{ij}E^3\,\gamma^3\epsilon_j - G\,\epsilon^i\,,\\
\delta E^3 &= -\,\bar\epsilon^i\gamma^3\hat{\partial\!\!\!/}\varphi^j\,\varepsilon_{ij} - \bar\epsilon_i\gamma^3\hat{\partial\!\!\!/}\,\varphi_j\,\varepsilon^{ij}\,,\\
\delta G &= -\,2\bar\epsilon_i\,\hat{\partial\!\!\!/}\varphi^i\,,
\end{aligned} \tag{3.5}$$

where $\hat{\partial\!\!\!/} \equiv \hat\gamma^{\hat\mu}\partial_{\hat\mu}$. The correct R-symmetry transformations can now be identified by adopting SU(2) transformations for the fermion fields φ^i, such that δL_{ij} and $\delta E_{\hat\mu 3}$ remain invariant under the combined transformations of the fermions and the supersymmetry parameters. This leads to

$$\delta\varphi^i = \tfrac12\mathrm{i}\alpha\,\varphi^i - \tfrac12\bar\beta\,\varepsilon^{ij}\gamma^3\,\varphi_j\,,\qquad \delta\varphi_i = -\tfrac12\mathrm{i}\alpha\,\varphi_i - \tfrac12\beta\,\varepsilon_{ij}\gamma^3\,\varphi^j\,. \tag{3.6}$$

The above transformations indeed generate the SU(2) group. It is then straightforward to establish that under this particular R-symmetry subgroup, the fields G, $\bar G$ and E^3 transform according to the vector representation,

$$\delta G = \mathrm{i}\alpha\,G - \bar\beta\,E^3\,,\qquad \delta E^3 = \tfrac12\beta\,G + \tfrac12\bar\beta\,\bar G\,. \tag{3.7}$$

Likewise, the dimensionally reduced supersymmetry transformations of the vector supermultiplet read,

$$\begin{aligned}
\delta X &= -\,\mathrm{i}\,\bar\epsilon^i\,\gamma^3\,\Omega_i\,,\\
\delta W_3 &= \mathrm{i}\,\bar\epsilon_i\,\Omega_j\,\varepsilon^{ij} - \mathrm{i}\,\bar\epsilon^i\,\Omega^j\,\varepsilon_{ij}\,,\\
\delta W_{\hat\mu} &= \bar\epsilon_i\hat\gamma_{\hat\mu}\,\Omega_j\,\varepsilon^{ij} + \bar\epsilon^i\hat\gamma_{\hat\mu}\,\Omega^j\,\varepsilon_{ij}\,,\\
\delta\Omega_i &= 2\mathrm{i}\,\hat{\partial\!\!\!/} X\,\gamma^3\,\epsilon_i + \mathrm{i}\hat{\partial\!\!\!/} W_3\,\varepsilon_{ij}\,\epsilon^j - \varepsilon_{ij}\,F^{\hat\mu}\hat\gamma_{\hat\mu}\epsilon^j + Y_{ij}\,\epsilon^j\,,\\
\delta Y_{ij} &= 2\,\bar\epsilon_{(i}\hat{\partial\!\!\!/}\,\Omega_{j)} + 2\,\varepsilon_{ik}\,\varepsilon_{jl}\,\bar\epsilon^{(k}\hat{\partial\!\!\!/}\,\Omega^{l)}\,.
\end{aligned} \tag{3.8}$$

where $F^{\hat\mu} = \frac12\mathrm{i}\varepsilon^{\hat\mu\hat\nu\hat\rho}\partial_{\hat\nu}W_{\hat\rho}$. The R-symmetry transformations of Ω^i follow from the invariance of $\delta W_{\hat\mu}$ and δY_{ij} under the combined transformations on the spinors. This time we find

$$\delta\Omega^i = \tfrac12\mathrm{i}\alpha\,\Omega^i + \tfrac12\bar\beta\,\varepsilon^{ij}\gamma^3\,\Omega_j\,,\qquad \delta\Omega_i = -\tfrac12\mathrm{i}\alpha\,\Omega_i + \tfrac12\beta\,\varepsilon_{ij}\gamma^3\,\Omega^j\,. \tag{3.9}$$

which also correctly generates the SU(2) group associated with (3.4). It then follows that the fields X, $\bar X$ and W_3 transform under the R-symmetry group according to the vector representation,

$$\delta X = -\mathrm{i}\alpha\,X + \tfrac12\beta\,W_3\,,\quad \delta W_3 = -\beta\,\bar X - \bar\beta\,X\,. \tag{3.10}$$

[2]Note that the Dirac conjugate of a spinor involves the matrix γ^0. Therefore there is a relative factor $\tilde\gamma$ between the three- and four-dimensional Dirac conjugates, and correspondingly between the two charge conjugation matrices. As a result the three-dimensional charge conjugation matrix $\hat C$ satisfies the following identities,

$$\hat C\hat\gamma^{\hat\mu}\hat C^{-1} = -\hat\gamma^{\hat\mu\mathrm{T}}\,,\quad \hat C\gamma^3\hat C^{-1} = \gamma^{3\mathrm{T}}\,,\quad \hat C\tilde\gamma\hat C^{-1} = \tilde\gamma^{\mathrm{T}}\,,\quad \hat C\gamma^5\hat C^{-1} = -\gamma^{5\mathrm{T}}\,,\quad \hat C^{\mathrm{T}} = -\hat C\,.$$

The above results enable the identification of the vector and tensor multiplet components, up to an overall constant and an SU(2) transformation that identifies the U(1) subgroup. To see this we write the spinor quantities for the tensor multiplet in a different basis. Following [29] we first write the supersymmetry parameters in a basis where the 'hidden' SU(2) factor of the R-symmetry becomes manifest,

$$\begin{aligned} \epsilon^+ &= \tfrac{1}{2}\sqrt{2}\,\gamma^3\,(\epsilon_1 - \mathrm{i}\epsilon_2)\,, \qquad & \epsilon^- &= \tfrac{1}{2}\sqrt{2}\,\left(\epsilon^1 - \mathrm{i}\epsilon^2\right)\,, \\ \epsilon_+ &= \tfrac{1}{2}\sqrt{2}\,\gamma^3\left(\epsilon^1 + \mathrm{i}\epsilon^2\right)\,, \qquad & \epsilon_- &= \tfrac{1}{2}\sqrt{2}\,(\epsilon_1 + \mathrm{i}\epsilon_2)\,. \end{aligned} \tag{3.11}$$

To appreciate this choice of basis we note that the SU(2) transformations (3.4) read

$$\delta\epsilon^+ = \tfrac{1}{2}\mathrm{i}(\alpha\,\epsilon^+ + \bar\beta\,\epsilon^-)\,. \tag{3.12}$$

Note that $\epsilon^\pm$ and $\epsilon_\pm$ are related through charge conjugation. Likewise we write the tensor multiplet spinors as,

$$\begin{aligned} \varphi^+ &= -\tfrac{1}{2}\sqrt{2}\,\gamma^3\,\left(\varphi^1 + \mathrm{i}\varphi^2\right)\,, \qquad & \varphi^- &= -\tfrac{1}{2}\sqrt{2}\,(\varphi_1 + \mathrm{i}\varphi_2)\,, \\ \varphi_+ &= -\tfrac{1}{2}\sqrt{2}\,\gamma^3\,(\varphi_1 - \mathrm{i}\varphi_2)\,, \qquad & \varphi_- &= -\tfrac{1}{2}\sqrt{2}\,\left(\varphi^1 - \mathrm{i}\varphi^2\right)\,, \end{aligned} \tag{3.13}$$

where the relevance of this basis follows from

$$\delta\varphi^+ = \tfrac{1}{2}\mathrm{i}(\alpha\,\varphi^+ + \bar\beta\,\varphi^-)\,. \tag{3.14}$$

In this basis the supersymmetry transformations of the tensor multiplet can be compared directly to those of the vector multiplet components, where we identify the spinor fields (φ^+, φ^-) with the spinor fields (Ω^1, Ω^2) of the vector multiplet. This establishes the c-map for the bosonic degrees of freedom,

$$\begin{aligned} &L_{12} = \mathrm{i}(X + \bar X)\,, \quad L_{11} = W_3 + X - \bar X\,, \quad L_{22} = W_3 - X + \bar X\,, \\ &E_{\hat\mu 3} = W_{\hat\mu}\,, \\ &G = Y_{22}\,, \quad \bar G = Y_{11}\,, \quad E^3 = \mathrm{i}Y_{12}\,. \end{aligned} \tag{3.15}$$

3.2 On higher-derivative actions

The expressions for the composite chiral supermultiplet can also be used to construct actions with higher-derivative couplings. For instance, we can start from the simple $N = 2$ supersymmetric Lagrangian for a single vector multiplet,

$$\mathcal{L} \propto |\partial_\mu X|^2 + \tfrac{1}{8}\,{F_{\mu\nu}}^2 + \tfrac{1}{2}\bar\Omega^i\,\partial\!\!\!/\,\Omega_i - \tfrac{1}{8}\,|Y_{ij}|^2, \tag{3.16}$$

and substitute the expressions for the composite components X, $F_{\mu\nu}$, Ω_i and Y_{ij} in terms of the tensor multiplet components. These are encoded in a function $\mathcal{F}(L)$ subject to the constraint,

$$\frac{\partial^2\mathcal{F}(L)}{\partial L^{ij}\,\partial L_{ij}} = 0\,. \tag{3.17}$$

This constraint enables one to show that the action depends only on a single function $\mathcal{H}(L) = [\mathcal{F}(L)]^2$ which is no longer subject to constraints. To demonstrate this we present the bosonic terms,

$$
\begin{aligned}
\mathcal{L} = {}& \mathcal{H}\Big[-\tfrac{1}{2}|\partial^2 L_{ij}|^2 + 2\partial_{[\mu}E_{\nu]}\,\partial^{[\mu}E^{\nu]} + |\partial_\mu G|^2\Big] \\
& + \mathcal{H}^{ij}\Big[\Big(\partial_{[\mu}L_{ik}\partial_{\nu]}L_{jl}\varepsilon^{kl}\Big)\partial^\mu E^\nu + 2\left(\partial_{[\mu}L_{ij}E_{\nu]}\left(\partial^\mu E^\nu\right)\right) - \tfrac{1}{2}E^2\partial^2 L_{ij} - |G|^2\,\partial^2 L_{ij} \\
& \qquad - \varepsilon_{ik}\big(E^\mu\partial_\mu L_{jl}\big)\big(\partial^2 L^{kl}\big) - \tfrac{1}{2}\big(\partial_\mu L_{ik}\,\partial^\mu L_{jl}\big)\,\partial^2 L^{kl}\Big] \\
& - \tfrac{1}{2}\mathcal{H}^{ij,kl}\Big[\big(\partial_\mu L_{ik}\,\partial^\mu L_{jl}\big)|G|^2 + \tfrac{1}{2}\varepsilon_{ik}\varepsilon_{jl}\big(|G|^2+E^2\big)^2 - 2\varepsilon_{ik}\big(E^\mu\partial_\mu L_{jl}\big)\big(|G|^2+E^2\big) \\
& \qquad - 2\varepsilon_{ik}\big(\partial_\mu L_{lp}\partial^\mu L^{np}\big)E^\nu\partial_\nu L_{jn} - \varepsilon_{ik}\varepsilon_{lm}E^\mu E^\nu\big(\partial_\nu L^{mn}\big)\big(\partial_\mu L_{jn}\big) \\
& \qquad - \varepsilon_{ik}\varepsilon_{jm}E^2\big(\partial_\mu L_{ln}\partial^\mu L^{mn}\big) + \tfrac{1}{2}\,\varepsilon_{ik}\,\partial_\mu L_{jm}\,\partial^\mu L^{pq}\,\partial_\nu L_{nq}\,\partial^\nu L_{lp}\,\varepsilon^{mn}\Big]\,,
\end{aligned}
\tag{3.18}
$$

where

$$
\mathcal{H}^{ij} = \frac{\partial\mathcal{H}}{\partial L_{ij}}\,, \qquad \mathcal{H}^{ij,kl} = \frac{\partial^2\mathcal{H}}{\partial L_{ij}\,\partial L_{kl}}\,. \tag{3.19}
$$

Let us make a few comments at this point. First of all, consider the linear combination of the free tensor multiplet Lagrangians (2.21) and (3.18),

$$
\begin{aligned}
\mathcal{L} = {}& -\tfrac{1}{2}|\partial_\mu L_{ij}|^2 + E_\mu\,E^\mu - (\bar\varphi^i\,\partial\!\!\!/\varphi_i + \bar\varphi_i\,\partial\!\!\!/\varphi^i) + |G|^2 \\
& + M^{-2}\Big[-\tfrac{1}{2}|\partial^2 L_{ij}|^2 + 2\partial_{[\mu}E_{\nu]}\,\partial^{[\mu}E^{\nu]} + (\partial^2\bar\varphi^i\,\partial\!\!\!/\varphi_i + \partial^2\bar\varphi_i\,\partial\!\!\!/\varphi^i) + |\partial_\mu G|^2\Big]\,,
\end{aligned}
\tag{3.20}
$$

where M is a mass parameter. This action describes a free massless tensor multiplet and a *massive* vector supermultiplet, as can be shown by analyzing the corresponding equations of motion. The massive multiplet corresponds to negative metric states. All of this is in accord with standard off-shell counting arguments.

Another comment concerns the R-symmetry. Lagrangians that are at most quadratic in derivatives are always invariant under one of the factors of the R-symmetry group, but not necessarily under both factors. For instance, the two-derivative action for vector multiplets is always invariant under the SU(2) R-symmetry subgroup but not necessarily under the U(1) factor. For the tensor multiplets the situation is precisely the reverse. In this respect the Lagrangians that depend quartically on derivatives are different as they can potentially break both factors of the R-symmetry group.

Although it is in principle possible to convert the tensor field to a scalar field by a duality transformation, the fact that the Lagrangian (3.18) contains quartic terms in E^μ and terms with derivatives of E^μ, makes it rather difficult to obtain explicit expressions.

Finally, in the following sections we will discuss the coupling of tensor supermultiplets to supergravity. In that context it is rather straightforward to also couple Lagrangians with higher derivatives to supergravity. As we do not intend to cover this topic in more detail here, we only present the supergravity coupling to the Lagrangian (3.18), restricting ourselves again to the purely bosonic terms. Such a coupling requires $\mathcal{H}(L)$ to be an SU(2) invariant function that is homogeneous of degree -2. The result can then be written as follows,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3\,, \tag{3.21}$$

where

$$\begin{aligned} e^{-1}\mathcal{L}_1 =& \mathcal{H}(L)\Big\{ -\tfrac{1}{2}L_{ij}L^{ij}\big(\tfrac{1}{3}R+D\big)^2 + \big(E^2 - L^{ij}\mathcal{D}^2 L_{ij}\big)\,\big(\tfrac{1}{3}R+D\big) + |G|^2\big(\tfrac{1}{6}R+2D\big) \\ & - \mathcal{D}_a E_b\big(R^{abi}{}_j(\mathcal{V})L_{ik}\varepsilon^{jk} - \tfrac{1}{2}[T^{abij}\varepsilon_{ij}\,G + \text{h.c.}]\big) \\ & + \tfrac{1}{8}\big(R_{ab}{}^i{}_j(\mathcal{V})L_{ik}\varepsilon^{jk} - \tfrac{1}{2}[T_{ab}{}^{ij}\varepsilon_{ij}\,G + \text{h.c.}]\big)^2 - \tfrac{1}{64}[T_{ab}{}^{ij}\varepsilon_{ij}G + \text{h.c.}]^2 \\ & + |\mathcal{D}_\mu G|^2 - \tfrac{1}{2}\,\big(\mathcal{D}^2 L_{ij}\big)\,\big(\mathcal{D}^2 L^{ij}\big) + 2\mathcal{D}_{[a}E_{b]}\,\mathcal{D}^{[a}E^{b]}\Big\}\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} e^{-1}\mathcal{L}_2 =& -\tfrac{1}{2}\mathcal{H}^{ij}(L)\Big\{L^{kl}\big(\mathcal{D}_\mu L_{ik}\mathcal{D}^\mu L_{jl}\big)\,\big(\tfrac{1}{3}R+D\big) \\ & + \big(E_b\mathcal{D}_a L_{ij} + \tfrac{1}{2}\mathcal{D}_a L_{ik}\mathcal{D}_b L_{jl}\varepsilon^{kl}\big)\big(R^{abm}{}_n(\mathcal{V})L_{mo}\varepsilon^{no} \\ & - \tfrac{1}{2}[T^{abmn}\varepsilon_{mn}G + \text{h.c.}]\big) \\ & - \mathcal{D}_\mu L_{ij}\mathcal{D}^\mu|G|^2 - 2(\mathcal{D}_a L_{ik}\mathcal{D}_b L_{jl}\varepsilon^{kl} - E_a\mathcal{D}_b L_{ij})(\mathcal{D}^a E^b) \\ & + \mathcal{D}^2 L_{ij}(|G|^2 + 2E^2) + \mathcal{D}^2 L^{kl}(\mathcal{D}_\mu L_{ik}\mathcal{D}^\mu L_{jl} + 2\varepsilon_{ik}E^\mu\mathcal{D}_\mu L_{jl})\Big\}\,, \end{aligned} \tag{3.23}$$

$$\begin{aligned} e^{-1}\mathcal{L}_3 =& \tfrac{1}{4}\mathcal{H}^{ij,kl}(L)\Big\{\varepsilon_{ik}\varepsilon_{jl}(\mathcal{D}_\mu L_{mn}\mathcal{D}^\mu L^{mn}|G|^2 - (|G|^2+E^2)^2 + \tfrac{1}{4}(\mathcal{D}_\mu L_{mn}\mathcal{D}^\mu L^{mn})^2) \\ & - 2(\mathcal{D}_\mu L_{ik}\mathcal{D}^\mu L_{jl})E^2 + 4\varepsilon_{ik}[E^\mu\mathcal{D}_\mu L_{jl}(|G|^2+E^2) \\ & - (\mathcal{D}_\mu L_{jm}\mathcal{D}_\nu L_{ln}\varepsilon^{mn})E^\mu E^\nu] \\ & - \tfrac{1}{2}\varepsilon_{ik}\varepsilon_{jl}(\mathcal{D}^\mu L_{mn}\mathcal{D}^\nu L^{mn})^2 - 2\varepsilon_{ik}E^\nu\mathcal{D}_\mu L_{jl}(\mathcal{D}_\mu L_{mn}\mathcal{D}^\mu L^{mn})\Big\}\,. \end{aligned} \tag{3.24}$$

Here R denotes the Ricci scalar associated with the gravitational field. The other supergravity fields will be introduced in the next section. Because this expression is based on a chiral superspace density, one can also introduce elementary vector multiplets as well as the Weyl multiplet couplings. The latter are accompanied by additional terms of higher order in the Riemann tensor. Terms such as these may be important for determining the subleading corrections to the black hole entropy [18].

4. Coupling to conformal supergravity

The tensor supermultiplet constitutes also a representation of the full $N = 2$ superconformal algebra [2]. In addition to the translations, Lorentz transformations, and R-symmetry transformations, the fields are subject to dilatations. In principle, fields also transform under conformal boosts, but matter multiplets are usually inert under those. On the fermionic side, the conventional Q-supersymmetry is extended with a second, special, supersymmetry, called S-supersymmetry.

Superconformal transformations can be defined in flat space, with space-time independent transformation parameters and transformation rules that explicitly depend on the space-time coordinates. In a superconformal background, where the translations are replaced by space-time diffeomorphisms, the transformation rules contain the various (gauge and other) fields of the superconformal theory. In their presence the Q- and S-supersymmetry transformations of the tensor supermultiplet fields take the following form,

$$\begin{aligned}
\delta L_{ij} &= 2\,\bar\epsilon_{(i}\varphi_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\varphi^{l)}\,,\\
\delta\varphi^i &= \not{D}L^{ij}\,\epsilon_j + \varepsilon^{ij}\,\hat{\not{E}}^I\,\epsilon_j - G\,\epsilon^i + 2L^{ij}\,\eta_j\,,\\
\delta G &= -\,2\,\bar\epsilon_i\not{D}\,\varphi^i\; - \bar\epsilon_i(6\,L^{ij}\,\chi_j + \tfrac14\,\gamma^{ab}T_{abjk}\,\varphi_l\,\varepsilon^{ij}\varepsilon^{kl}) + 2\,\bar\eta_i\varphi^i\,,\\
\delta E_{\mu\nu} &= \mathrm{i}\bar\epsilon^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij} - \mathrm{i}\bar\epsilon_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij} \;+\; 2\mathrm{i}\,L_{ij}\,\varepsilon^{jk}\,\bar\epsilon^i\gamma_{[\mu}\psi_{\nu]k} - 2\mathrm{i}\,L^{ij}\,\varepsilon_{jk}\,\bar\epsilon_i\gamma_{[\mu}\psi_{\nu]}{}^k\,.
\end{aligned} \tag{4.1}$$

Here ϵ^i and η^i denote the Q- and S-supersymmetry parameters, respectively. The derivatives D_μ are superconformally covariant and $\hat E^\mu$ denotes the superconformally covariant field strength of the tensor field $E_{\mu\nu}$. These quantities, which will be defined shortly, involve the gauge fields of the superconformal algebra: the dilatational gauge field b_μ, the U(1) and SU(2) R-symmetry gauge fields A_μ and $\mathcal{V}_\mu{}^i{}_j$, the spin connection field $\omega_\mu{}^{ab}$, the gauge field $f_\mu{}^a$ associated with special conformal boosts, and the Q- and S-supersymmetry gauge fields $\psi_\mu{}^i$ and $\phi_\mu{}^i$. Not all of these gauge fields are independent and we refer to the appendix for further details. Obviously, we also have the vierbein field $e_\mu{}^a$ and its inverse which are used to convert world to tangent space indices and vice versa. Apart from the gauge fields, the superconformal theory contains a complex, anti-selfdual tensor field $T_{ab}{}^{ij}$, a spinor field χ^i and a real scalar field D, of which only the first two appear in (4.1).

To exhibit some of the details we record the expressions for the superconformal derivatives and the superconformal tensor field strength,

$$\begin{aligned}
D_\mu L_{ij} &= \mathcal{D}_\mu L_{ij} - \bar\psi_{\mu(i}\,\varphi_{j)} - \varepsilon_{ik}\varepsilon_{jl}\,\bar\psi_\mu^{(k}\,\varphi^{l)}\,,\\
D_\mu\varphi^i &= \mathcal{D}_\mu\varphi^i - L^{ij}\,\phi_{\mu j} - \tfrac12\left(\not{D}L^{ij} + \varepsilon^{ij}\,\hat{\not{E}}\right)\psi_{\mu j} + \tfrac12 G\,\psi_\mu{}^i\,,\\
D_\mu G &= \mathcal{D}_\mu G - \bar\phi_{\mu i}\varphi^i + \bar\psi_{\mu\, i}\,\not{D}\varphi^i + 3\,L^{ij}\,\bar\psi_{\mu\, i}\chi_j + \tfrac18\,\bar\psi_{\mu i}\,\gamma^{cd}\,T_{cd\,kl}\,\varphi_j\,\varepsilon^{ik}\,\varepsilon^{lj}\,,\\
\hat E^\mu &= \tfrac12\mathrm{i}\,e^{-1}\,\varepsilon^{\mu\nu\rho\sigma}\Big[\partial_\nu E_{\rho\sigma} - \tfrac12\mathrm{i}\bar\psi^i_\nu\gamma_{\rho\sigma}\varphi^j\varepsilon_{ij} + \tfrac12\mathrm{i}\bar\psi_{\nu i}\gamma_{\rho\sigma}\varphi_j\varepsilon^{ij} - \mathrm{i}\,L_{ij}\varepsilon^{jk}\bar\psi_\nu{}^i\gamma_\rho\psi_{\sigma k}\Big]\,.
\end{aligned} \tag{4.2}$$

Here the derivatives $\mathcal{D}_\mu$ are covariant with respect to Lorentz transformations, dilatations and R-symmetry transformations,

$$\begin{aligned}
\mathcal{D}_\mu L_{ij} &= (\partial_\mu - 2b_\mu)L_{ij} - \mathcal{V}_\mu{}^k{}_{(i}\,L_{j)k}\\
\mathcal{D}_\mu\varphi^i &= (\partial_\mu - \tfrac14\omega_\mu{}^{ab}\,\gamma_{ab} - \tfrac12\mathrm{i}A_\mu - \tfrac52 b_\mu)\varphi^i + \tfrac12\mathcal{V}_\mu{}^i{}_j\,\varphi^j\,,\\
\mathcal{D}_\mu G &= (\partial_\mu - \mathrm{i}\,A_\mu - 3\,b_\mu)G\,.
\end{aligned} \tag{4.3}$$

The coupling between a tensor and a reduced chiral supermultiplet is still possible in a superconformal background [3]. The reduced chiral multiplet constitutes also a superconformal multiplet and transforms under Q- and S-supersymmetry according to

$$\begin{aligned}
\delta X &= \bar\epsilon^i\Omega_i\,,\\
\delta\Omega_i &= 2\,\not{D}X\,\epsilon_i + \tfrac12\varepsilon_{ij}\gamma^{\mu\nu}\hat F_{\mu\nu}\,\epsilon^j + Y_{ij}\,\epsilon^j + 2X\,\eta_i\,,\\
\delta Y_{ij} &= 2\,\bar\epsilon_{(i}\,\not{D}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\,\not{D}\Omega^{l)}\,.
\end{aligned} \tag{4.4}$$

Here we have introduced a superconformal field strength $\hat F_{\mu\nu}$, defined by

$$\begin{aligned}\hat F_{\mu\nu} =&\ F_{\mu\nu} - \bar\psi_{[\mu i}\gamma_{\nu]}\Omega_j\,\varepsilon^{ij} - \bar\psi_{[\mu}{}^i\gamma_{\nu]}\Omega^j\,\varepsilon_{ij}\\ &- X\,\bar\psi_{\mu i}\psi_{\nu j}\,\varepsilon^{ij} - \bar X\,\bar\psi_\mu{}^i\psi_\nu{}^j\,\varepsilon_{ij} - \tfrac14\bar X\,T_{\mu\nu}{}^{ij}\varepsilon_{ij} - \tfrac14 X\,T_{\mu\nu ij}\varepsilon^{ij}\,,\end{aligned} \tag{4.5}$$

where $F_{\mu\nu} = 2\,\partial_{[\mu}W_{\nu]}$. The supersymmetry variation of W_μ remains as given in (2.3). The superconformal field strength should be identified with a component of the superconformal reduced chiral multiplet, as can be seen from its variation under Q- and S-supersymmetry,

$$\delta\hat F^-_{ab} = \tfrac12\bar\epsilon_i\not{D}\gamma_{ab}\Omega_j\,\varepsilon^{ij} - \tfrac12\bar\epsilon^i\gamma_{ab}\not{D}\Omega^j\,\varepsilon_{ij} - \bar\eta_i\gamma_{ab}\Omega_j\,\varepsilon^{ij}\,. \tag{4.6}$$

The superconformally invariant coupling between the two multiplets is an extension of (2.4),

$$\begin{aligned}e^{-1}\mathcal{L} =&\ X\,G + \bar X\,\bar G - \tfrac12 Y^{ij}\,L_{ij}\\ &-\tfrac12(\bar\psi_\mu{}^i\gamma^\mu\Omega^j + \bar X\,\bar\psi_\mu{}^i\gamma^{\mu\nu}\psi_\nu{}^j)\,L_{ij} - \tfrac12(\bar\psi_{\mu i}\gamma^\mu\Omega_j + X\,\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j})\,L^{ij}\\ &+\bar\varphi^i(\Omega_i + X\,\gamma^\mu\psi_{\mu i}) + \bar\varphi_i(\Omega^i + \bar X\,\gamma^\mu\psi_\mu{}^i)\\ &-\tfrac14\mathrm{i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,E_{\mu\nu}\,F_{\rho\sigma}\,.\end{aligned} \tag{4.7}$$

Just as in the previous section, we can construct reduced chiral multiplets from tensor multiplets. Again we start with the complex scalar X_I defined in (2.5), which transforms into a chiral spinor $\Omega_{i\,I}$,

$$\begin{aligned}X_I =&\ \mathcal{F}_{I,J}\,\bar G^J + \mathcal{F}_{I,JK}{}^{ij}\,\bar\varphi_i{}^J\varphi_j{}^K\,,\\ \Omega_{i\,I} =&\ -2\,\mathcal{F}_{I,J}\,\not{D}\varphi_i{}^J - \mathcal{F}_{I,J}(6\,L_{ij}{}^J\chi^j + \tfrac14 T_{ab}{}^{jk}\,\gamma^{ab}\varphi^{lJ}\,\varepsilon_{ij}\varepsilon_{kl}) + 2\,\mathcal{F}_{I,JKij}\,\bar G^J\,\varphi^{jK}\\ &-2\,\mathcal{F}_{I,JK}{}^{kl}\,(\not{D}L_{ik}{}^J - \varepsilon_{ik}\not{\hat E}^J)\varphi_l{}^K + 2\,\mathcal{F}_{I,JKLij}{}^{kl}\,\varphi^{jL}\,(\bar\varphi_k{}^J\varphi_l{}^K)\,.\end{aligned} \tag{4.8}$$

Here the function $\mathcal{F}_{I,J}(L)$ should again satisfy the constraints (2.10). But in order that (4.8) defines the beginning of a superconformal reduced chiral multiplet, the component X_I should, in addition, be invariant under S-supersymmetry. This is precisely ensured by the condition (2.32), which implies that the function $\mathcal{F}_{I,J}$ is SU(2) invariant and homogeneous of degree -1, so that it has scaling weight -2. As it turns out, there are no further restrictions and we simply record the corresponding expressions for $Y_{ij\,I}$ and $F_{\mu\nu\,I}$ below,

$$\begin{aligned}Y_{ij\,I} =&\ -2\,\mathcal{F}_{I,J}\left[\Box^{\mathrm{c}}L_{ij}{}^J + 3\,DL_{ij}{}^J\right] - 2\,\mathcal{F}_{I,JKij}\,(\bar G^J\,G^K + \hat E_\mu{}^J\,\hat E^{\mu K})\,,\\ &-2\,\mathcal{F}_{I,JK}{}^{kl}\,(D_\mu L_{ik}{}^J\,D^\mu L_{jl}{}^K + 2\,\varepsilon_{k(i}\,D_\mu L_{j)l}{}^J\,\hat E^{\mu K})\\ &-2\,\mathcal{F}_{I,JKLij}{}^{kl}\,\bar\varphi_k{}^K\varphi_l{}^J\,G^L - 2\,\mathcal{F}_{I,JKLijkl}\,\bar\varphi^{kK}\varphi^{lJ}\,\bar G^L\\ &+4\,(\mathcal{F}_{I,JKm(i}\,\bar\varphi^{mJ}\,\not{D}\varphi_{j)}{}^K + \mathcal{F}_{I,JK}{}^{m(k}\bar\varphi_m{}^J\,\not{D}\varphi^{l)K}\,\varepsilon_{ik}\,\varepsilon_{jl})\\ &+4\,\mathcal{F}_{I,JKLn(i}{}^{kl}\,D_\mu L_{j)k}{}^J\,\left(\bar\varphi^{nL}\gamma^\mu\varphi_l{}^K\right)\\ &-4\,\mathcal{F}_{I,JKLn(i}{}^{kl}\,\varepsilon_{j)k}\,\left(\bar\varphi^{nL}\,\not{\hat E}^J\,\varphi_l{}^K\right)\\ &-2\,\mathcal{F}_{I,JKLMijmn}{}^{kl}\,\bar\varphi_k{}^J\varphi_l{}^K\,\bar\varphi^{mL}\varphi^{nM}\\ &+12\,\mathcal{F}_{I,JKk(i}\,L_{j)l}{}^J(\bar\varphi^{kK}\chi^l + \varepsilon^{km}\varepsilon^{ln}\bar\varphi_m{}^K\chi_n)\end{aligned}$$

$$+\tfrac{1}{2}\mathcal{F}_{I,JKk(i}(\varepsilon_{j)m}\,\bar\varphi^{kJ}\gamma^{ab}T_{ab}{}^{mn}\varphi^{lK}\,\varepsilon_{nl}+\varepsilon^{mk}\,\bar\varphi_m{}^J\gamma^{ab}T_{ab\,j)n}\varphi_l{}^K\,\varepsilon^{nl})\,, \tag{4.9}$$

$$\begin{aligned} F_{\mu\nu\,I} \;=\; & -2\,\mathcal{F}_{I,JK}{}^{mn}\,\partial_{[\mu}L_{mk}{}^J\,\partial_{\nu]}L_{nl}{}^K\,\varepsilon^{kl} \\ & -4\,\partial_{[\mu}\Big(\mathcal{F}_{I,J}\,\hat E_{\nu]}{}^J+\mathcal{F}_{I,JKki}\,\bar\varphi^{kJ}\gamma_{\nu]}\varphi_j{}^K\,\varepsilon^{ij}\Big) \\ & +2\,\partial_{[\mu}\Big(\mathcal{F}_{I,J}\,\mathcal{V}_{\nu]}{}^i{}_j\,L_{ik}{}^J\,\varepsilon^{jk}+\mathcal{F}_{I,J}\,\bar\psi_{\nu]}{}^i\,\varphi^{jJ}\,\varepsilon_{ij}+\mathcal{F}_{I,J}\,\bar\psi_{\nu]i}\,\varphi_j{}^J\,\varepsilon^{ij}\Big)\,, \end{aligned} \tag{4.10}$$

where $\Box^{\rm c}L_{ij}=D^aD_aL_{ij}$. An explicit evaluation leads to the following expression,

$$\begin{aligned} D_a\,D^a\,L_{ij}{}^I = \;& \mathcal{D}_aD^aL_{ij}{}^I+2\,f_\mu{}^\mu\,L_{ij}{}^I \\ & -\Big(\bar\psi^\mu{}_{(i}\,D_\mu\,\varphi_{j)}{}^I+\varepsilon_{ik}\,\varepsilon_{jl}\,\bar\psi^{\mu(k}\,D_\mu\varphi^{l)I}\Big) \\ & -\tfrac{1}{16}\,\Big(\bar\varphi_{(j}{}^I\,T^{cd}{}_{i)k}\,\gamma_{cd}\,\gamma^\mu\,\psi^k_\mu+\varepsilon_{ik}\,\varepsilon_{jl}\,\bar\varphi^{(lI}\,T^{k)mcd}\,\gamma_{cd}\,\gamma^\mu\,\psi_{\mu m}\Big) \\ & -\tfrac{3}{2}\,\Big(\bar\psi_{\mu(i}\,L_{j)k}{}^I\,\gamma^\mu\,\chi^k-\bar\psi_\mu{}^k\,\gamma^\mu\,L_{k(i}{}^I\,\chi_{j)}+\bar\psi_\mu{}^k\,\gamma^\mu\,\chi_k\,L_{ij}{}^I\Big) \\ & +\tfrac{1}{2}\,\Big(\bar\phi_{\mu(i}\,\gamma^\mu\,\varphi_{j)}{}^I+\varepsilon_{ik}\,\varepsilon_{jl}\,\bar\phi_\mu{}^{(k}\,\gamma^\mu\,\varphi^{l)I}\Big)\,. \end{aligned} \tag{4.11}$$

Obviously the equations (4.8), (4.9) and (4.10) are extensions of the expressions (2.5), (2.11) and (2.12). For a single tensor supermultiplet the results can be compared to [3]. We also note that it is possible to recast the superconformal extension $\hat F_{abI}$ of (4.10) in a form where its supercovariance is more manifest,

$$\begin{aligned} \hat F_{ab\,I} = \;& -4\,D_{[a}\,\Big(\mathcal{F}_{I,J}\,\hat E^J_{b]}\Big)-2\,\mathcal{F}_{I,JKij}\,D_{[a}\,L^{ikJ}\,D_{b]}L^{jlK}\,\varepsilon_{kl}+\mathcal{F}_{I,J}\,R_{ab}{}^i{}_j(\mathcal{V})\,L^J_{ik}\,\varepsilon^{jk} \\ & -\mathcal{F}_{I,J}\,\bar\varphi^{iJ}\,R_{ab}{}^j(Q)\,\varepsilon_{ij}-\tfrac14\,T_{ab}{}^{ij}\,\varepsilon_{ij}\,\Big(\mathcal{F}_{I,J}\,G^J+\mathcal{F}_{I,JKkl}\,\bar\varphi^{kJ}\,\varphi^{lK}\Big) \\ & -\mathcal{F}_{I,J}\,\bar\varphi_i{}^J\,R_{abj}(Q)\,\varepsilon^{ij}-\tfrac14\,T_{abij}\,\varepsilon^{ij}\,\Big(\mathcal{F}_{I,J}\,\bar G^J+\mathcal{F}_{I,JK}{}^{kl}\,\bar\varphi_k{}^J\,\varphi_l{}^K\Big) \\ & -D_{[a}\,\Big(4\,\mathcal{F}_{I,JKij}\,\bar\varphi^{iK}\,\gamma_{b]}\,\varphi_k{}^J\,\varepsilon^{jk}\Big)\,. \end{aligned} \tag{4.12}$$

However, to derive the Lagrangian it is much more convenient to work with the expression (4.10).

We now proceed and substitute the above expressions into the supercovariant density formula (4.7). In principle this is straightforward. In doing this we make use of the condition (2.32) and in order to express the Lagrangian in terms of a single function, we also use (2.33). Dropping a total derivative term, we then establish that the Lagrangian depends only on the function F_{IJ} that we encountered earlier in (2.23). The complete result can then be presented as follows,

$$\mathcal{L}_{\rm total}=e\mathcal{L}_1+e\mathcal{L}_2+e\mathcal{L}_3+e\mathcal{L}_4\,, \tag{4.13}$$

where

$$
\begin{aligned}
\mathcal{L}_1 = & F_{IJ}\, L_{ij}{}^{I}\, L^{ijJ} \Big\{ \tfrac{1}{3}\Big[R + (e^{-1}\varepsilon^{\mu\nu\rho\sigma} \bar\psi_\mu{}^i \gamma_\nu \mathcal{D}_\rho \psi_{\sigma i} - \tfrac{1}{4} \bar\psi_\mu{}^i \psi_\nu{}^j\, T^{\mu\nu}{}_{ij} + \text{h.c.})\Big] \\
& \qquad + D + \tfrac{1}{2}(\bar\psi_\mu{}^i \gamma^\mu \chi_i + \text{h.c.}) \Big\} , \\
\mathcal{L}_2 = & F_{IJ} \Big\{ -\tfrac{1}{2}\, \mathcal{D}_\mu L_{ij}{}^{I}\, \mathcal{D}^\mu L^{ijJ} + E_\mu{}^{I}\, E^{\mu J} - \bar\varphi^{iI}\, \not{\mathcal{D}} \varphi_i{}^{J} - \bar\varphi_i{}^{I}\, \not{\mathcal{D}} \varphi^{iJ} + G^I \bar G^J \\
& \qquad - \Big[(\tfrac{1}{8} \bar\varphi^{iI} \gamma_{\mu\nu} \varphi^{lJ}\, \varepsilon_{ij}\varepsilon_{kl} - \tfrac{1}{3} L_{ij}{}^{I}\, \bar\varphi^{iJ} \gamma_\mu \psi_{\nu k})\, T^{\mu\nu jk} + \text{h.c.} \Big] \\
& \qquad - \Big[L_{ij}{}^{I} (\tfrac{4}{3} \bar\varphi^{iJ} \gamma^{\mu\nu} \mathcal{D}_\mu \psi_\nu{}^j + 2 \bar\varphi^{iJ} \chi^j) + \text{h.c.} \Big] \\
& \qquad + \tfrac{1}{2}\Big[\bar\psi_{\mu i}\, [(\not{D} + \not{\mathcal{D}}) L^{ijI} - \varepsilon^{ij} (\hat{\not{E}}^I + \not{E}^I)]\, \gamma^\mu \varphi_j{}^{J} + \text{h.c.} \Big] \\
& \qquad + e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar\psi_\nu{}^i \gamma_\rho\, \psi_{\sigma k}\, L_{ij}{}^{I} \mathcal{D}_\mu L^{jkJ} - E^{\mu I}\, \mathcal{V}_\mu{}^i{}_j\, L_{ik}{}^{J}\, \varepsilon^{jk} , \\
& \qquad + e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar\psi_\nu{}^i \gamma_\rho\, \psi_{\sigma k}\, L_{ij}{}^{I}\, \varepsilon^{jk} \Big[E_\mu{}^{J} - \tfrac{1}{4} \bar\psi_\lambda{}^m \gamma_\mu \gamma^\lambda \varphi^{nJ} \varepsilon_{mn} - \tfrac{1}{4} \bar\psi_{\lambda m} \gamma_\mu \gamma^\lambda \varphi_n{}^{J} \varepsilon^{mn} \Big] \\
& \qquad + \tfrac{1}{4} \Big(e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar\psi_\nu{}^i \gamma_\rho\, \psi_{\sigma k}\, L_{ij}{}^{I}\, \varepsilon^{jk} \Big) \Big(e^{-1} \varepsilon_\mu{}^{\lambda\tau\zeta} \bar\psi_\lambda{}^m \gamma_\tau\, \psi_\zeta\, L_{mn}{}^{J}\, \varepsilon^{np} \Big) \Big\} , \\
\mathcal{L}_3 = & \tfrac{1}{2} \mathrm{i} e^{-1} \varepsilon^{\mu\nu\rho\sigma}\, F_{IJK}{}^{ij}\, E_{\mu\nu}{}^{I}\, \partial_\rho L_{ik}{}^{J}\, \partial_\sigma L_{jl}{}^{K}\, \varepsilon^{kl} \\
& + \Big\{ F_{IJKij} \Big(\bar G^I\, \bar\varphi^{iJ}\, \varphi^{jK} + \bar\varphi^{iI}\, (\not{D} L^{jkJ} + \varepsilon^{jk}\, \hat{\not{E}}^J)\, \varphi_k{}^{K} \\
& \qquad - \bar\varphi^{iI} \varphi^{jJ}\, \bar\psi_\mu{}^k \gamma^\mu \varphi_k{}^{K} - \bar\psi_\mu{}^i \varphi^{jI}\, \bar\varphi^{kJ} \gamma^\mu \varphi_k{}^{K} \Big) + \text{h.c.} \Big\} , \\
\mathcal{L}_4 = & F_{IJKLij}{}^{kl}\, \bar\varphi^{iI} \varphi^{jJ}\, \bar\varphi_k{}^{K} \varphi_l{}^{L} .
\end{aligned}
\tag{4.14}
$$

Setting the fields of the Weyl multiplet to zero, one recovers the tensor multiplet Lagrangian (2.21). For a single tensor supermultiplet the above expression may be compared to the result derived in [3].

5. Poincaré supergravity with tensor multiplets

Superconformal matter multiplets coupled to conformal supergravity are gauge equivalent to matter-coupled Poincaré supergravity provided that enough potential compensating multiplets are present. One compensating vector multiplet is needed to provide the graviphoton of $N = 2$ Poincaré supergravity. For the minimal off-shell versions one may choose a so-called non-linear multiplet, a hypermultiplet or a tensor multiplet. In the Poincaré context, the conformal symmetries (*i.e.*, scale transformations, special conformal boosts and S-supersymmetry) are no longer present. The R-symmetries are usually absent as well. An exception is the case where a single tensor multiplet acts as a compensator, because the triplet field L^{ij} has a U(1) stability subgroup which reflects itself as a local invariance group of the corresponding Poincaré supergravity Lagrangian [3]. The presence of other multiplets can nevertheless affect this local invariance, as we shall see in due course.

In the first subsection we focus on some characteristic features of the Poincaré supergravity Lagrangians with tensor multiplets. As already explained in section 1, it is important to stress our treatment is based on off-shell multiplets, as it is always possible to dualize tensor multiplets into hypermultiplets and, in the presence of suitable isometries,

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vice versa. This conversion affects, however, the off-shell structure of the theory. In a second subsection 5.2 we explain the structure of the tensor multiplet target space. In a third subsection 5.3 we work out the example of two tensor multiplets which, upon dualization, leads to the classification of 4-dimensional quaternion-Kähler manifolds with two abelian isometries. These manifolds include the so-called universal hypermultiplet which emerges in Calabi-Yau compactifications of string theory.

5.1 The general case

In this subsection we discuss the coupling of tensor, vector and hypermultiplets to supergravity. We first present the Lagrangians in their superconformally invariant form and exhibit a number of characteristic features that are relevant in the context of the super-Poincaré formulation. The coupling to tensor multiplets is based on this paper. For the hypermultiplets we follow the treatment of [7] and for the vector multiplets we base ourselves on [30, 31] and related references. In all three cases n will denote the number of independent multiplets. Of course, these numbers do not have to be equal, but we refrain from introducing extra notation to make a distinction. As it turns out the couplings for each of the three types of multiplets can be defined in terms of certain homogeneous potentials, which we denote by $\chi_{\text{tensor}}(L)$, $\chi_{\text{hyper}}(\phi)$ and $\chi_{\text{vector}}(X,\bar{X})$, respectively. Under the scale transformations of the superconformal group, these potentials scale with weight 2 and they are invariant under the R-symmetry group. As a result of the scale invariance, the target spaces parametrized by the scalar fields of each of the three supermultiplets are cones. For hypermultiplets the target space is a hyperkähler cone, which is a cone over a $(4n-1)$-dimensional 3-Sasakian space. The latter is an Sp(1) fibration over a $(4n-4)$-dimensional quaternion-Kähler space. The target space of the vector multiplets is a cone over the product of an $(2n-1)$-dimensional special Kähler space times S^1. The target space of the tensor multiplet is the cone over a $(3n-1)$-dimensional space whose geometrical properties have not been extensively studied so far.

In the case of tensor and vector multiplets, supersymmetry relates the gauge fields (*i.e.*, the tensor and vector fields) to a special basis for the scalar fields given by $L_{ij}{}^I$ and X^Λ, respectively. The potentials can therefore be generally defined in terms of these fields. Eventually $L_{ij}{}^I$ and X^Λ may be parametrized in terms of other fields, in which case they will play the role of sections. The case of hypermultiplets is different in this respect, because these multiplets do not contain any gauge fields and have thus no preferred basis for the scalars. Moreover hypermultiplets do not constitute off-shell supermultiplets, unlike the tensor and vector supermultiplets. For superconformal hypermultiplets there exists the so-called hyperkähler potential $\chi_{\text{hyper}}(\phi)$ [7], where the fields ϕ^A denote the $4n$ scalar fields corresponding to n hypermultiplets, but there is no a priori definition of the hyperkähler potential. The fact that we are dealing with hyperkähler cones implies that the derivative of $\chi_{\text{hyper}}(\phi)$ is directly related to a homothetic vector denoted by k^A,

$$\frac{\partial\chi_{\text{hyper}}(\phi)}{\partial\phi^A} = g_{AB}(\phi)\, k^B(\phi)\,. \tag{5.1}$$

Here $k = k^A\,\partial/\partial\phi^A$ generates the scale transformations on the target space of the hypermultiplet scalars and $g_{AB}(\phi)$ denotes the metric on the hyperkähler cone. We are dealing with an exact homothety, implying that

$$D_A k^B = \delta_A{}^B \quad \Leftrightarrow \quad D_A D_B \chi_{\rm hyper} = g_{AB}\,, \quad \chi_{\rm hyper}(\phi) = \tfrac{1}{2} g_{AB}\, k^A k^B\,. \tag{5.2}$$

The covariant derivative contains the Levi-Civitá connection associated with the metric g_{AB}. The formulation of the action and transformation rules for hypermultiplets is not determined exclusively in terms of the hyperkähler potential, and we note the existence of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ bundle [32] which appear naturally in the full Lagrangian and transformation rules (here n denotes the number of hypermultiplets and $\alpha = 1,\ldots,2n$). Here $\mathrm{Sp}(1)$ coincides with the SU(2) factor of the R-symmetry group and the $\mathrm{Sp}(n)$ group acts on the negative-chirality spinors ζ^α through the indices α. Indices referring to the conjugate $\mathrm{Sp}(n)$ representation will be denoted by $\bar\alpha$ and they label the positive-chirality spinors $\zeta^{\bar\alpha}$. Under S-supersymmetry the fermions transform into the sections $A_i{}^\alpha$ mentioned above.

For the three types of supermultiplets, the potentials are defined by

$$\begin{aligned}
\chi_{\rm tensor}(L) &= 2\,F_{IJ}\,L_{ij}{}^I\,L^{ijJ}\,,\\
\chi_{\rm hyper}(L) &= \tfrac{1}{2}\varepsilon^{ij}\bar\Omega_{\alpha\beta}\,A_i{}^\alpha A_j{}^\beta\,,\\
\chi_{\rm vector}(X,\bar X) &= \mathrm{i}\,\big(X^\Lambda \bar F_\Lambda - \bar X^\Lambda F_\Lambda\big) = N_{\Lambda\Sigma}\,X^\Lambda \bar X^\Sigma\,.
\end{aligned} \tag{5.3}$$

Here F_Λ is the derivative of a holomorphic homogeneous function $F(X)$ of second degree of the fields X^Λ and $N_{\Lambda\Sigma}$ is defined by

$$N_{\Lambda\Sigma} = \frac{\partial^2\chi_{\rm vector}(X,\bar X)}{\partial X^\Lambda \partial\bar X^\Sigma} = 2\,\mathrm{Im}[F_{\Lambda\Sigma}]\,, \tag{5.4}$$

where $F_{\Lambda\Sigma}$ denotes the second derivative of $F(X)$. Furthermore the symplectic tensor $\bar\Omega_{\alpha\beta}$, which exists for any hyperkähler space, can be defined as follows,

$$\bar\Omega_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{ij}\,g_{AB}\,\gamma^{Ai}{}_\alpha\,\gamma^{Bj}{}_\beta\,, \tag{5.5}$$

where $\gamma^{Ai}{}_\alpha$ denotes a generalized vielbein that converts hyperkähler target-space indices into $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ indices. This quantity appears in the supersymmetry transformations of the hypermultiplet scalars,

$$\delta\phi^A = 2\,(\gamma^A{}_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \gamma^{Ai}{}_\alpha\,\bar\epsilon_i\zeta^\alpha)\,. \tag{5.6}$$

The presentation above shows that the combined target space is a product of three cones, each with its own potential. The potentials satisfy properties that are very similar to (5.1) and (5.2), except that there is no need to use covariant derivatives as in (5.2). For tensor multiplets the corresponding equations are given by (2.37), (2.38) and (2.39). The homogeneity of the potentials follows from

$$L_{ki}{}^I\,\frac{\partial\chi_{\rm tensor}(L)}{\partial L_{kj}{}^I} = \tfrac{1}{2}\delta^j{}_i\,\chi_{\rm tensor}(L)\,,$$

$$
\begin{aligned}
k^A(\phi)\,\frac{\partial\chi_{\rm hyper}(\phi)}{\partial\phi^A} &= 2\,\chi_{\rm hyper}(\phi)\,,\\
X^\Lambda\,\frac{\partial\chi_{\rm vector}(X,\bar X)}{\partial X^\Lambda} = \bar X^\Lambda\,\frac{\partial\chi_{\rm vector}(X,\bar X)}{\partial\bar X^\Lambda} &= \chi_{\rm vector}(X,\bar X)\,.
\end{aligned}
\tag{5.7}
$$

For the tensor and vector multiplet potentials the above equations also imply the invariance under R-symmetry. For the hyperkähler potential the equations for R-symmetry involve the relevant Killing vectors, or the complex structures of the hyperkähler cone.

Let us now exhibit some characteristic terms of the three Lagrangians, and compare them (eventually we also consider the sum of the three Lagrangians),

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm tensor} =&\ \tfrac{1}{6}\,\chi_{\rm tensor}\Big[R+(e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i}-\tfrac14\bar\psi_\mu{}^i\psi_\nu{}^j\,T^{\mu\nu}{}_{ij}+\text{h.c.})\Big]\\
&+\tfrac12\,\chi_{\rm tensor}\Big[D+\tfrac12(\bar\psi_\mu{}^i\gamma^\mu\chi_i+\text{h.c.})\Big]\,,\\
&-\tfrac12 F_{IJ}\,\mathcal{D}_\mu L_{ij}{}^I\,\mathcal{D}^\mu L^{ijJ}\\
&-\Big(\frac{\partial\chi_{\rm tensor}}{\partial L^{ijI}}\Big[\tfrac23\bar\varphi^{iI}\gamma^{\mu\nu}\mathcal{D}_\mu\psi_\nu{}^j+\bar\varphi^{iI}\chi^j-\tfrac16\,\bar\varphi^{iI}\gamma_\mu\psi_{\nu k}\,T^{\mu\nu jk}\Big]+\text{h.c.}\Big)\,,\\
e^{-1}\mathcal{L}_{\rm hyper} =&\ \tfrac16\,\chi_{\rm hyper}\Big[R+(e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i}-\tfrac14\bar\psi_\mu{}^i\psi_\nu{}^j\,T^{\mu\nu}{}_{ij}+\text{h.c.})\Big]\\
&+\tfrac12\,\chi_{\rm hyper}\Big[D+\tfrac12(\bar\psi_\mu{}^i\gamma^\mu\chi_i+\text{h.c.})\Big]\,,\\
&-\tfrac12 g_{AB}\,\mathcal{D}_\mu\phi^A\,\mathcal{D}^\mu\phi^B\\
&-\frac{\partial\chi_{\rm hyper}}{\partial\phi^A}\Big(\gamma^A{}_{i\bar\alpha}\Big[\tfrac23\bar\zeta^{\bar\alpha}\gamma^{\mu\nu}\mathcal{D}_\mu\psi_\nu{}^i+\bar\zeta^{\bar\alpha}\chi^i-\tfrac16\,\bar\zeta^{\bar\alpha}\gamma_\mu\psi_{\nu j}\,T^{\mu\nu ij}\Big]+\text{h.c.}\Big)\,,\\
e^{-1}\mathcal{L}_{\rm vector} =&\ \tfrac16\chi_{\rm vector}\Big[R+(e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i}+\tfrac12\bar\psi_\mu{}^i\psi_\nu{}^j\,T^{\mu\nu}{}_{ij}+\text{h.c.})\Big]\\
&-\chi_{\rm vector}\Big[D+\tfrac12(\bar\psi_\mu{}^i\gamma^\mu\chi_i+\text{h.c.})\Big]\,,\\
&-N_{\Lambda\Sigma}\,\mathcal{D}_\mu X^\Lambda\,\mathcal{D}^\mu\bar X^\Sigma\\
&-\Big(\frac{\partial\chi_{\rm vector}}{\partial X^\Lambda}\Big[\tfrac13\bar\Omega_i{}^\Lambda\gamma^{\mu\nu}\mathcal{D}_\mu\psi_\nu{}^i-\bar\Omega_i{}^\Lambda\chi^i+\tfrac16\bar\Omega_i{}^\Lambda\gamma_\mu\psi_{\nu j}\,T^{\mu\nu ij}\Big]+\text{h.c.}\Big)\,.
\end{aligned}
\tag{5.8}
$$

The equations (5.8) exhibit a rather uniform structure for the various couplings. Especially the couplings of tensor multiplets and hypermultiplets are closely related, which is not surprising in view of the fact that the tensor multiplets can be dualized to hypermultiplets. The fact that the potentials for the tensor multiplet cones and the hyperkähler cones are identical, a result derived at the end of subsection 2.3, makes the agreement even more close.

With the vector multiplet there are subtle differences reflected in the relative coefficients. It is well known that these differences are crucial for converting to the Poincaré formulation. The above Lagrangians still contain gauge degrees of freedom associated with certain superconformal symmetries. The symmetry under conformal boosts is manifest. Because only the dilatational gauge field b_μ transforms under this symmetry, it follows that the Lagrangians are independent of b_μ, as can be verified by explicit computation. The dilatational symmetry is still intact and we can impose a corresponding gauge condition. The obvious condition is to set the coefficient of the Ricci scalar in the combined Lagrangian to a constant, *i.e.*,

$$\tfrac{1}{6}\chi_{\text{tensor}} + \tfrac{1}{6}\chi_{\text{hyper}} + \tfrac{1}{6}\chi_{\text{vector}} = -\frac{1}{2\,\kappa^2}\,, \tag{5.9}$$

so that we end up with a conventional Einstein-Hilbert term. Observe that, in order to describe scalar fields with kinetic terms of the correct sign, it follows that the cone metrics can not be positive definite. Under Q-supersymmetry the condition (5.9) is not invariant, and it is convenient to exploit S-supersymmetry to set its variation to zero by a second gauge choice. This motivates the condition,

$$2\,\frac{\partial\chi_{\text{tensor}}}{\partial L^{ijI}}\,\varphi^{jI} + 2\,\frac{\partial\chi_{\text{hyper}}}{\partial\phi^A}\gamma^A{}_{i\bar\alpha}\,\zeta^{\bar\alpha} + \frac{\partial\chi_{\text{vector}}}{\partial X^\Lambda}\,\Omega_i{}^\Lambda = 0\,. \tag{5.10}$$

Concentrating on the second and fourth lines of the three Lagrangians (5.8) we see that the fields D and χ^i act as Lagrange multipliers, which leads, when combined with the above gauge choices, to the following results,

$$\begin{aligned} \chi_{\text{tensor}} + \chi_{\text{hyper}} &= -2\,\kappa^{-2}\,,\\ \chi_{\text{vector}} &= -\kappa^{-2}\,,\\ \frac{\partial\chi_{\text{tensor}}}{\partial L^{ijI}}\,\varphi^{jI} + \frac{\partial\chi_{\text{hyper}}}{\partial\phi^A}\,\gamma^A{}_{i\bar\alpha}\,\zeta^{\bar\alpha} &= 0\,,\\ \frac{\partial\chi_{\text{vector}}}{\partial X^\Lambda}\,\Omega_i{}^\Lambda &= 0\,. \end{aligned} \tag{5.11}$$

Because we used the field equations corresponding to one bosonic and eight fermionic fields, supersymmetry is no longer realized off shell. As a result of the above procedure the sum of the Lagrangians (5.8) reduces to

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{combined}} = {} & -\frac{1}{2\,\kappa^2}R - \frac{1}{2\,\kappa^2}\Big[e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \tfrac{1}{4}\bar\psi_\mu{}^i\psi_\nu{}^j\,T^{\mu\nu}{}_{ij} + \text{h.c.}\Big]\\ & - \tfrac{1}{2}F_{IJ}\,\mathcal{D}_\mu L_{ij}{}^I\,\mathcal{D}^\mu L^{ijJ} - \tfrac{1}{2}g_{AB}\,\mathcal{D}_\mu\phi^A\,\mathcal{D}^\mu\phi^B - N_{\Lambda\Sigma}\,\mathcal{D}_\mu X^\Lambda\,\mathcal{D}^\mu\bar X^\Sigma\,. \end{aligned} \tag{5.12}$$

In this formulation the scalar fields are constrained by the first two equations of (5.11) and corresponding restrictions exist on the fermions. The full action is now invariant under general coordinate transformations, local Lorentz transformations, local supersymmetry (defined as a field-dependent linear combination of Q- and S-supersymmetry) and local R-symmetry. Clearly the hypermultiplet and tensor multiplet fields are entangled whereas the vector multiplet fields remain separate, a feature that has been known for some time.

5.2 The tensor multiplet target space

We now specialize to the tensor scalars $L_{ij}{}^I$ and analyze their corresponding target space. It is convenient to change notation at this point and rescale the fields $L_{ij}{}^I$ by the inverse of χ_{tensor} so that the $L_{ij}{}^I$ are scale invariant (we refrain from imposing a gauge condition). The rescaled fields are then constrained to a hypersurface,

$$2\,F_{IJ}(L)\,L_{ij}{}^I L^{ijJ} = 1\,. \tag{5.13}$$

Furthermore we use a vector notation for the fields $L_{ij}{}^I$, according to

$$L_{ij}{}^I = -\mathrm{i}\,\vec{L}^I \cdot (\vec{\sigma})^k{}_i\,\varepsilon_{jk}\,, \tag{5.14}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices (with $\sigma_1\sigma_2\sigma_3 = \mathrm{i}$) so that $L_{ij}{}^I L^{ijJ} = 2\,\vec{L}^I \cdot \vec{L}^J$. With these definitions we find,

$$\tfrac{1}{2} F_{IJ}(L)\,\mathcal{D}_\mu L_{ij}{}^I \mathcal{D}^\mu L_{ij}{}^I = \frac{(\partial_\mu \chi_{\text{tensor}})^2}{4\,\chi_{\text{tensor}}} + \chi_{\text{tensor}}\,F_{IJ}(L)\,\mathcal{D}_\mu \vec{L}^I \cdot \mathcal{D}^\mu \vec{L}^J\,, \tag{5.15}$$

which shows that we are indeed dealing with a cone over a $(3n-1)$-dimensional space parametrized by the constrained coordinates $\vec{L}^I$.

Let us now write the bosonic terms of the Lagrangian (4.14) in terms of the rescaled variables,

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{tensor}} = {} & \chi_{\text{tensor}} \Big[\tfrac{1}{6}R + \tfrac{1}{2}D - \tfrac{1}{4}(\partial_\mu \ln \chi_{\text{tensor}})^2\Big] \\ & - \chi_{\text{tensor}} F_{IJ}(L)\,(\partial_\mu \vec{L}^I - \vec{\mathcal{V}}_\mu \times \vec{L}^I) \cdot (\partial^\mu \vec{L}^J - \vec{\mathcal{V}}^\mu \times \vec{L}^J) \\ & + \chi_{\text{tensor}}^{-1} F_{IJ}(L) \Big[E_\mu{}^I\,E^{\mu J} + G^I \bar{G}^J\Big] \\ & + 2\,F_{IJ}(L)\,E^{\mu I}\,\vec{L}^J\,\vec{\mathcal{V}}_\mu - \tfrac{1}{2}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\vec{F}_{IJK}(L) \cdot (\partial_\rho \vec{L}^I \times \partial_\sigma \vec{L}^J)\,E_{\mu\nu}{}^K\,, \end{aligned} \tag{5.16}$$

where $\vec{F}_{IJK} = \partial F_{IJ}/\partial \vec{L}^K$ and

$$\mathcal{V}_\mu{}^i{}_j = \mathrm{i}\vec{\mathcal{V}}_\mu\,(\vec{\sigma})^i{}_j\,. \tag{5.17}$$

To eliminate the auxiliary SU(2) gauge fields $\vec{\mathcal{V}}_\mu$, the matrix that multiplies the terms quadratic in these fields is relevant,

$$[\mathbf{M}]^{rs} = F_{IJ}\,\vec{L}^I \cdot \vec{L}^J\,\delta^{rs} - L^{Ir}\,F_{IJ}\,L^{Js}\,, \tag{5.18}$$

where $r, s = 1, 2, 3$ denote vector indices. It is clear that this matrix has zero eigenvalues whenever all vectors $\vec{L}^I$ are aligned, which is related to the fact that these configurations leave a subgroup of SU(2) invariant. This is especially relevant for the case of a single tensor multiplet, which always leaves a subgroup invariant, so that the approach sketched below is not applicable. For several tensor multiplets generic configurations correspond to matrices $\mathbf{M}$ with non-vanishing determinant. In that case the equations of motion for $\vec{\mathcal{V}}_\mu$ lead to

$$\vec{\mathcal{V}}_\mu = \mathbf{M}^{-1}\,(\vec{L}^I \times \partial_\mu \vec{L}^J + \chi_{\text{tensor}}^{-1}\,E_\mu{}^I\,\vec{L}^J)\,F_{IJ}\,, \tag{5.19}$$

where the inverse $\mathbf{M}^{-1}$ equals[3]

$$\left[\mathbf{M}^{-1}\right]_{rs} = \frac{1}{\det(\mathbf{M})}\Big[\tfrac{1}{2}F_{IJ}F_{KL}(\vec{L}^I\times\vec{L}^K)\cdot(\vec{L}^J\times\vec{L}^L)\,\delta_{rs} + L_r{}^I\,F_{IK}(\vec{L}^K\cdot\vec{L}^L)F_{LJ}\,L_s{}^J\Big]\,. \tag{5.20}$$

The determinant of $\mathbf{M}$ is given by

$$\det(\mathbf{M}) = \tfrac{1}{3}(F_{IJ}\vec{L}^I\cdot\vec{L}^J)^3 - \tfrac{1}{3}F_{IJ}(\vec{L}^J\cdot\vec{L}^K)F_{KL}(\vec{L}^L\cdot\vec{L}^M)F_{MN}(\vec{L}^N\cdot\vec{L}^I)\,. \tag{5.21}$$

The Lagrangian (5.16) is invariant under tensor gauge transformations, up to a surface term. The latter originates exclusively from the last term in (5.16). To establish this one needs to use the condition

$$\frac{\partial}{\partial\vec{L}^I}\cdot\frac{\partial}{\partial\vec{L}^J}\,F_{KL} = 0\,, \tag{5.22}$$

which follows from (2.38), and which was extensively discussed in section 2. Under local SU(2) transformations the Lagrangian is also invariant up to a surface term. These transformations can be written as

$$\delta\vec{L}^I = \vec{\Lambda}\times\vec{L}^I\,,\qquad \delta\vec{\mathcal{V}} = \partial_\mu\vec{\Lambda} + \vec{\Lambda}\times\vec{\mathcal{V}}\,, \tag{5.23}$$

where $\vec{\Lambda}(x)$ represents the infinitesimal space-time dependent parameters of SU(2), and the variation of the Lagrangian (resulting from the last two terms in (5.16)) reads,

$$\delta_\Lambda\mathcal{L}_{\text{tensor}} = \partial_\mu\Big(-\,\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\,F_{IJ}\,\vec{L}^I\cdot\partial_\nu\vec{\Lambda}\,E_{\rho\sigma}{}^J\Big)\,. \tag{5.24}$$

Substituting (5.19) in the Lagrangian (5.16) then leads to the following Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{tensor}} &= \chi_{\text{tensor}}\Big[\tfrac{1}{6}R + \tfrac{1}{2}D - \tfrac{1}{4}(\partial_\mu\ln\chi_{\text{tensor}})^2\Big] \\ &\quad - \chi_{\text{tensor}}\,\Big[\mathcal{G}^{(1)}_{IJ}\,\partial_\mu\vec{L}^I\,\partial^\mu\vec{L}^J + \mathcal{G}^{(2)}_{IJ,KL}\,(\vec{L}^I\cdot\partial_\mu\vec{L}^J)\,(\vec{L}^K\cdot\partial_\mu\vec{L}^L)\Big] \\ &\quad + \chi_{\text{tensor}}^{-1}\,\Big[\mathcal{H}^{(1)}_{IJ}\,E_\mu{}^I\,E^{\mu J} + F_{IJ}\,G^I\bar{G}^J\Big] \\ &\quad + E^{\mu I}\,\mathcal{H}^{(2)}_{IJ}\,\vec{L}^J\cdot(\vec{L}^K\times\partial_\mu\vec{L}^L)\,F_{KL} \\ &\quad - \tfrac{1}{2}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\vec{F}_{IJK}\cdot(\partial_\rho\vec{L}^I\times\partial_\sigma\vec{L}^J)\,E_{\mu\nu}{}^K\,, \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} \mathcal{H}^{(1)}_{IJ} &= F_{IJ} + F_{IK}\,L_r{}^K\,(\mathbf{M}^{-1})^{rs}\,L_s{}^L\,F_{LJ}\,, \\ \mathcal{H}^{(2)}_{IJ} &= \frac{1}{\det(\mathbf{M})}\Big[[(F_{KL}\vec{L}^K\cdot\vec{L}^L)^2 - F_{KL}\vec{L}^L\cdot\vec{L}^M F_{MN}\vec{L}^N\cdot\vec{L}^K]\,F_{IJ} \\ &\qquad + 2\,F_{IK}\vec{L}^K\cdot\vec{L}^L F_{LM}\vec{L}^M\cdot\vec{L}^N F_{NJ}\Big]\,, \end{aligned}$$

[3]Direct verification of this result makes use of the identity for general 3×3 matrices $\mathcal{O}$,

$$\mathcal{O}^3 - \mathrm{tr}(\mathcal{O})\,\mathcal{O}^2 + \tfrac{1}{2}[(\mathrm{tr}(\mathcal{O}))^2 - \mathrm{tr}(\mathcal{O}^2)]\,\mathcal{O} = \det(\mathcal{O})\,\mathbf{1}\,.$$

$$
\begin{aligned}
\mathcal{G}^{(1)}_{IJ} &= F_{IJ} - \frac{(F_{KL}\vec{L}^K\cdot\vec{L}^L)^2 + F_{KL}\vec{L}^L\cdot\vec{L}^M F_{MN}\vec{L}^N\cdot\vec{L}^K}{2\,\det(\mathbf{M})}\, F_{IP}\vec{L}^P\cdot\vec{L}^Q F_{QJ} \\
&\quad + \frac{1}{\det(\mathbf{M})} F_{IK}\vec{L}^K\cdot\vec{L}^L F_{LM}\vec{L}^M\cdot\vec{L}^N F_{NP}\vec{L}^P\cdot\vec{L}^Q F_{QJ}\,, \\
\mathcal{G}^{(2)}_{IJ,KL} &= \frac{1}{\det(\mathbf{M})}\, F_{IM}\vec{L}^M\cdot\vec{L}^N F_{NK}\, F_{JP}\vec{L}^P\cdot\vec{L}^Q F_{QL} \\
&\quad + \frac{(F_{KL}\vec{L}^K\cdot\vec{L}^L)^2 + F_{KL}\vec{L}^L\cdot\vec{L}^M F_{MN}\vec{L}^N\cdot\vec{L}^K}{2\,\det(\mathbf{M})}\, F_{IL}\, F_{JK} \\
&\quad - \frac{1}{\det(\mathbf{M})}\Big[F_{IM}\vec{L}^M\cdot\vec{L}^N F_{NP}\vec{L}^P\cdot\vec{L}^Q F_{QL}\, F_{JK} + (I\leftrightarrow K; J\leftrightarrow L)\Big]\,. \qquad (5.26)
\end{aligned}
$$

The elimination of the SU(2) gauge fields does not affect the invariance under local SU(2). This means that the target space involves only $3(n-1)$ scalar fields, subject to the constraint (5.13), which in the present notation reads,

$$F_{IJ}\,\vec{L}^I\cdot\vec{L}^J = \tfrac{1}{4}\,. \qquad (5.27)$$

In principle one can now construct the most general variety of these spaces, starting from the (homogeneous) potential $\chi_{\rm tensor}$ written in terms of SU(2) invariant variables. Subsequently one imposes the conditions (2.37) and (2.38), which yield a number of second-order differential equations. Every solution of these equations yields a corresponding Lagrangian. Finally one imposes the constraint (5.27). At this point one has the option to convert the tensor fields $E_{\mu\nu}{}^I$ to scalars and obtain a quaternion-Kähler manifold of dimensions $4(n-1)$ with n abelian isometries. We already mentioned that the case $n=1$ is special, and also the cases $n=2$ and 3 are rather specific. The $n>3$ cases can be dealt with in a more generic way. In the next subsection we demonstrate this procedure for the case of $n=2$ tensor multiplets. In this way we will rather conveniently obtain the classification of 4-dimensional quaternion-Kähler spaces with two commuting isometries presented in [9]. We intend to return to an analysis of the higher-n cases in the future.

5.3 The case of two tensor supermultiplets

To illustrate the procedure sketched in the previous subsection we start by considering the most general potential $\chi_{\rm tensor}$ for two tensor multiplets, $\vec{L}^1$ and $\vec{L}^2$. This potential must be invariant under SU(2) rotations and homogeneous of first degree under a uniform rescaling of the $L_{ij}{}^I$. In order to incorporate these constraints it is convenient to introduce the SU(2) invariant variables,

$$s = \vec{L}^1\cdot\vec{L}^1\,, \qquad u = \frac{(\vec{L}^1\cdot\vec{L}^1)\,(\vec{L}^2\cdot\vec{L}^2) - (\vec{L}^1\cdot\vec{L}^2)^2}{s^2}\,, \qquad v = \frac{\vec{L}^1\cdot\vec{L}^2}{s}\,. \qquad (5.28)$$

Note that $s,u\geq 0$ and that u vanishes whenever the two vectors $\vec{L}^1$ and $\vec{L}^2$ are aligned. For $u=0$ we thus expect singularities as this value corresponds to field configurations that are invariant under a subgroup of SU(2). When expressed in terms of the above variables, the most general potential must be of the form

$$\chi_{\rm tensor} = \sqrt{2s}\, f(u,v)\,. \qquad (5.29)$$

Substituting this ansatz into (2.37) determines the entries of the matrix F_{IJ} to be

$$F_{IJ} = \frac{1}{\sqrt{2s}} \begin{pmatrix} \frac{1}{2}f - vf_v - uf_u + v^2 f_u & \frac{1}{2}f_v - vf_u \\ \frac{1}{2}f_v - vf_u & f_u \end{pmatrix} . \tag{5.30}$$

We also need the 2×2 matrix

$$\vec{L}^I \cdot \vec{L}^J = s \begin{pmatrix} 1 & v \\ v & u + v^2 \end{pmatrix} . \tag{5.31}$$

Imposing the constraint (2.38) leads to the following partial differential equation for $f(u, v)$,

$$f_{vv} + 4u\, f_{uu} = 0 \,. \tag{5.32}$$

Thus the most general Lagrangian for two tensor multiplets coupled to supergravity is based on the potential (5.29) with the function $f(u, v)$ subject to (5.32). In passing, we note the perturbatively corrected hypermultiplet [33, 34] corresponds to the following expression for the underlying tensor multiplet potential,

$$\chi_{\text{tensor}} = -2\sqrt{s}(u + 2c) \,, \tag{5.33}$$

which indeed satisfies the differential equation (5.32). Here the constant c is determined by the one-loop string correction to the universal hypermultiplet.

For a small number of tensor multiplets the various terms in the bosonic Lagrangian are most conveniently obtained from (5.16). Since we already established the invariance under local SU(2) we can consider a special gauge. In principle the SO(3) vector space can be decomposed into the two-dimensional space spanned by the vectors $\vec{L}^I$ and a one-dimensional subspace orthogonal to it. By adopting a gauge condition one can ensure that the derivatives $\partial_\mu \vec{L}^I$ take their values in the subspace spanned by the $\vec{L}^I$. With this condition one derives that $\vec{L}^I \cdot (\vec{L}^J \times \vec{L}^K) = 0$, and this suffices to show that the last term of (5.16) vanishes. Likewise, upon substituting the expression (5.19) for the SU(2) gauge field, all terms linear in $E^{\mu I}$ vanish as well for the same reason. Note that the above considerations pertain specifically to the case $n = 2$.

Now let us be more explicit about the gauge choice. By an appropriate rotation we can bring the $\vec{L}^I$ in the form

$$\vec{L}^1 = (\sqrt{s}, 0, 0) \,, \qquad \vec{L}^2 = (v\sqrt{s}, \sqrt{su}, 0) \,, \tag{5.34}$$

so that their inner products satisfy (5.31). Because the SU(2) is local we can ensure that this decomposition holds for all space-time points, so that the $\partial_\mu \vec{L}^I$ can be obtained consistently from (5.34). It is now easy to evaluate the matrix $\mathbf{M}$ defined in (5.18), which has a block-diagonal decomposition,

$$\mathbf{M} = \sqrt{\frac{s}{8}} \begin{pmatrix} Q_{2\times 2} & 0 \\ 0 & f \end{pmatrix} , \tag{5.35}$$

with the 2×2 matrix Q defined by

$$Q = \begin{pmatrix} 2uf_u & -\sqrt{u}f_v \\ -\sqrt{u}f_v & f - 2uf_u \end{pmatrix} . \tag{5.36}$$

The fields $\vec{\mathcal{V}}_\mu$ can now be evaluated explicitly and substituted into the Lagrangian. This leads to the following kinetic terms for the scalar fields s, u and v,

$$\begin{aligned} \mathcal{L}_{\rm scalars} = -\frac{e\,\chi_{\rm tensor}}{\sqrt{2s}} \Big[\frac{f}{8s}\,(\partial_\mu s)^2 + \frac{1}{2}\partial_\mu s\,\partial^\mu f + \frac{s\,f_u}{4u}\Big((\partial_\mu u)^2 + 4u(\partial_\mu v)^2\Big) \\ - \frac{s}{8uf}(f_v\,\partial_\mu u - uf_u\,\partial_\mu v)^2\Big] . \end{aligned} \tag{5.37}$$

Taking into account the fact that the three fields s, u, v are constrained by (5.27), which implies

$$s = \frac{1}{2\,f^2} , \tag{5.38}$$

one directly establishes

$$\begin{aligned} e^{-1}\mathcal{L}_{n=2} = {}& \chi_{\rm tensor}\Big[\tfrac{1}{6}R + \tfrac{1}{2}D - \tfrac{1}{4}(\partial_\mu \ln \chi_{\rm tensor})^2\Big] \\ & - \frac{\chi_{\rm tensor}\,\det(Q)}{(4\,uf)^2}\,\Big[(\partial_\mu u)^2 + 4u\,(\partial_\mu v)^2\Big] \\ & + \chi_{\rm tensor}^{-1}\,\Big[\mathcal{H}^{(1)}_{IJ}\,E_\mu{}^I\,E^{\mu J} + F_{IJ}\,G^I\bar{G}^J\Big] , \end{aligned} \tag{5.39}$$

where

$$\mathcal{H}^{(1)}_{IJ} = \frac{f\,\det(Q)}{u\,\sqrt{8\,s}}\,[N\,Q^{-2}N^{\rm T}]_{IJ} . \tag{5.40}$$

Here s is determined by (5.38) and the matrix N is defined by

$$N = \begin{pmatrix} \sqrt{u} & -v \\ 0 & 1 \end{pmatrix} . \tag{5.41}$$

It is straightforward to perform a duality transformation by introducing Lagrange multipliers ϕ_I to impose the Bianchi identity on the field strengths $E^{\mu I}$ and by subsequently integrating out the field strengths. The resulting line element is then equal to (we suppress the overall factor $\chi_{\rm tensor}$),

$$\mathrm{d}s^2 = \frac{\det(Q)}{(4\,uf)^2}\,[\mathrm{d}u^2 + 4u\,\mathrm{d}v^2] + [\mathcal{H}^{(1)}]^{IJ}\,\mathrm{d}\phi_I\,\mathrm{d}\phi_J , \tag{5.42}$$

where $[\mathcal{H}^{(1)}]^{IJ}$ is the inverse of (5.40). Upon a change of coordinates this line element coincides precisely with the expression derived by Calderbank and Pedersen for the general class of selfdual Einstein metrics with two commuting Killing fields [9]. In this work the matrices Q and N are related but not quite identical to the matrices used above. Hence, the formalism discussed in this paper enables a straightforward and elegant derivation of this classification.

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A. Superconformal calculus

Throughout this paper we use Pauli-Källén conventions and follow the notation used *e.g.* in [35]. Space-time indices are denoted by $\mu, \nu, \ldots$ and Lorentz indices by $a, b, \ldots$. Furthermore SU(2)-indices are denoted by $i, j, \ldots$ and the corresponding SO(3)-indices by $r, s, \ldots$. All (anti-)symmetrizations are with unit strength. Majorana spinors are defined by $\bar{\varphi} = \varphi^T C$, where the four-dimensional charge conjugation matrix C satisfies

$$-\gamma_\mu^{\mathrm{T}} = C\,\gamma_\mu\,C^{-1}\,, \qquad \gamma_5^{\mathrm{T}} = C\,\gamma_5\,C^{-1}\,, \qquad C^{\mathrm{T}} = -C\,. \tag{A.1}$$

The superconformal algebra consists of general coordinate, local Lorentz, dilatation, special conformal, chiral U(1) and SU(2), and Q- and S-supersymmetry transformations. Under Q-supersymmetry, S-supersymmetry and conformal transformations the independent fields of the Weyl multiplet transform as follows:

$$\begin{aligned}
\delta e_\mu{}^a &= \bar{\epsilon}^i\,\gamma^a \psi_{i\mu} + \bar{\epsilon}_i\,\gamma^a \psi_\mu^i\,,\\
\delta \psi_\mu{}^i &= 2\,\mathcal{D}_\mu \epsilon^i - \tfrac{1}{8} T_{ab}{}^{ij} \gamma^{ab} \gamma_\mu \epsilon_j - \gamma_\mu \eta^i\,,\\
\delta b_\mu &= \tfrac{1}{2} \bar{\epsilon}^i \phi_{\mu i} - \tfrac{3}{4} \bar{\epsilon}^i \gamma_\mu \chi_i - \tfrac{1}{2} \bar{\eta}^i \psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a}\,,\\
\delta A_\mu &= \tfrac{1}{2} i \bar{\epsilon}^i \phi_{\mu i} + \tfrac{3}{4} i \bar{\epsilon}^i \gamma_\mu\,\chi_i + \tfrac{1}{2} i \bar{\eta}^i \psi_{\mu i} + \text{h.c.}\,,\\
\delta \mathcal{V}_\mu{}^i{}_j &= 2\,\bar{\epsilon}_j \phi_\mu{}^i - 3 \bar{\epsilon}_j \gamma_\mu\,\chi^i + 2 \bar{\eta}_j\,\psi_\mu{}^i - (\text{h.c. ; traceless})\,,\\
\delta T_{ab}{}^{ij} &= 8\,\bar{\epsilon}^{[i} R_{ab}{}^{j]}(Q)\,,\\
\delta \chi^i &= -\tfrac{1}{12} \gamma^{ab}\,\not{D} T_{ab}{}^{ij}\,\epsilon_j + \tfrac{1}{6} R(\mathcal{V})_{\mu\nu}{}^i{}_j \gamma^{\mu\nu} \epsilon^j - \tfrac{1}{3} \mathrm{i} R_{\mu\nu}(A) \gamma^{\mu\nu} \epsilon^i + D \epsilon^i + \tfrac{1}{12} \gamma_{ab} T^{abij} \eta_j\,,\\
\delta D &= \bar{\epsilon}^i\,\not{D} \chi_i + \bar{\epsilon}_i\,\not{D} \chi^i\,.
\end{aligned} \tag{A.2}$$

Here Λ_K^a is the transformation parameter for conformal transformations. The full superconformally covariant derivative is denoted by D_μ while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations, *e.g.*, (also see eqs. (4.2) and (4.3))

$$\mathcal{D}_\mu \epsilon^i = \Big(\partial_\mu - \tfrac{1}{4} \omega_\mu{}^{cd}\,\gamma_{cd} + \tfrac{1}{2}\,b_\mu - \tfrac{1}{2} \mathrm{i}\,A_\mu\Big)\,\epsilon^i + \tfrac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{A.3}$$

	Weyl multiplet											parameters	
field	$e_\mu{}^a$	ψ_μ^i	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	T_{ab}^{ij}	χ^i	D	ω_μ^{ab}	$f_\mu{}^a$	ϕ_μ^i	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		+					+				$-$	+	$-$

Table 1: Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

	Tensor multiplet				
field	$E_{\mu\nu}$	L^{ij}	φ_i	G	F_{IJ}
w	0	2	$\frac{5}{2}$	3	-2
c	0	0	$-\frac{1}{2}$	1	0
γ_5			$-$		

Table 2: Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the tensor multiplet component fields.

	Vector multiplet			
field	X	Ω_i	W_μ	Y_{ij}
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0
γ_5		+		

Table 3: Weyl and chiral weights (w and c, respectively) and fermion chirality (γ_5) of the vector multiplet component fields.

The supercovariant curvature tensors[4] used here as well as in the main part of the paper are defined as

$$\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}\,e_{\nu]}{}^a + 2\,b_{[\mu}\,e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\,e_{\nu]b} - \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R_{\mu\nu}{}^i(Q) &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{8}\,T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R_{\mu\nu}(A) &= 2\,\partial_{[\mu}A_{\nu]} - i\left(\tfrac{1}{2}\bar\psi_\mu{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \text{h.c.}\right)\,,\\
R_{\mu\nu}{}^i{}_j(\mathcal{V}) &= 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j + 2(\bar\psi_{[\mu}{}^i\,\phi_{\nu]j} - \bar\psi_{i[\mu}\,\phi_{\nu]}{}^j) - 3(\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_j - \bar\psi_{[\mu j}\gamma_{\nu]}\chi^i)\\
&\quad - \delta_j{}^i(\bar\psi_{[\mu}{}^k\,\phi_{\nu]k} - \bar\psi_{[\mu k}\,\phi_{\nu]}{}^k) + \tfrac{3}{2}\delta_j{}^i(\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_k - \bar\psi_{[\mu k}\gamma_{\nu]}\chi^k)\,,\\
R_{\mu\nu}{}^{ab}(M) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&\quad + (\tfrac{1}{2}\bar\psi_{[\mu}{}^i\,T_{ij}^{ab}\,\phi_{\nu]}{}^j - \tfrac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R^{ab}{}_i(Q) + \text{h.c.})\,,\\
R_{\mu\nu}(D) &= 2\,\partial_{[\mu}b_{\nu]} - 2f_{[\mu}{}^a e_{\nu]a} - \tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \tfrac{1}{2}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + \tfrac{3}{4}\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,.
\end{aligned} \tag{A.4}$$

[4]We corrected a typo in [35] in the definition of $R_{\mu\nu}(D)$.

The remaining curvature tensors, $R_{\mu\nu}{}^i(S)$ and $R_{\mu\nu}{}^a(K)$, are not needed here, but may be found in [35]. There are three conventional constraints,

$$\begin{aligned}
&R_{\mu\nu}(P) = 0\,,\\
&\gamma^\mu\left(R_{\mu\nu}(Q)^i + \tfrac{1}{2}\gamma_{\mu\nu}\chi^i\right) = 0\,,\\
&e^\nu{}_b\,R_{\mu\nu}(M)_a{}^b - \mathrm{i}\tilde{R}_{\mu a}(A) + \tfrac{1}{8}T_{abij}T_\mu{}^{bij} - \tfrac{3}{2}D\,e_{\mu a} = 0\,,
\end{aligned} \tag{A.5}$$

which determine the fields $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$. We only used the expressions,

$$\begin{aligned}
\phi_\mu{}^i &= \tfrac{1}{2}\left(\gamma^{\rho\sigma}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\gamma^{\rho\sigma}\right)\left(\mathcal{D}_\rho\psi_\sigma{}^i - \tfrac{1}{16}T^{abij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^i\right)\,,\\
f_\mu{}^\mu &= \tfrac{1}{6}R - D - \left(\tfrac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \tfrac{1}{12}\bar{\psi}_\mu{}^i\psi_\nu{}^jT^{\mu\nu}{}_{ij} - \tfrac{1}{4}\bar{\psi}_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\right)\,.
\end{aligned} \tag{A.6}$$

When combining the conventional constraints with the various Bianchi identities one establishes that the curvatures are not all independent. For instance we note the relation,

$$\tilde{R}_{\mu\nu}(D) - \mathrm{i}R_{\mu\nu}(A) = 0\,. \tag{A.7}$$

For convenience, the Weyl and chiral weights together with the chirality of the spinors belonging to the Weyl, tensor and vector multiplet, are summarized in the tables 1, 2 and 3, respectively.

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